SCOTT, FORESMAN EXPERIMENTAL MATHEMATICS PROJECT

BOOKLET

7

The System of Natural Numbers

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EXPLORING IDEAS
Lesson 1
Booklet 7

In this lesson you will learn how members of one set can be matched with members of another set.

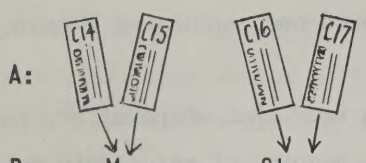
Time out for a chat

You already know some things about the natural numbers from your study of arithmetic. In earlier lessons you worked with the members of the set of natural numbers. You used natural numbers as replacements for variables in conditions, and you solved problems about natural numbers. You also worked with rate pairs in which the components were natural numbers. Finally, you learned about different numeration systems for naming the natural numbers.

In this booklet you are going to learn more about the natural numbers themselves. You will learn how they are associated with sets. Then you will learn more about the operations with natural numbers and the properties of the operations. Finally, you will learn what is meant by a number system.

A Tom has four concert tickets to sell. Suppose that he sells two tickets to Mary. How many tickets could he sell to Steve? Display 1 shows one way to think about this problem.

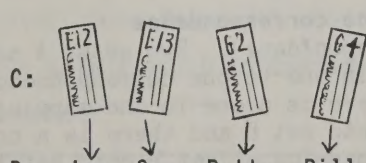
1



B Display 1 shows one way to match the members of A with the members of B. Is each member of A matched with just one member of B? Is each member of B used in the matching?

C Ed also sold four tickets. Display 2 shows the tickets that Ed sold and the names of the students to whom he sold them. Ed sold how many tickets to Joe?

2



D Have we matched each member of the set of tickets that Ed sold with just one member of the set of students? Is each member of the set of students used in the matching?

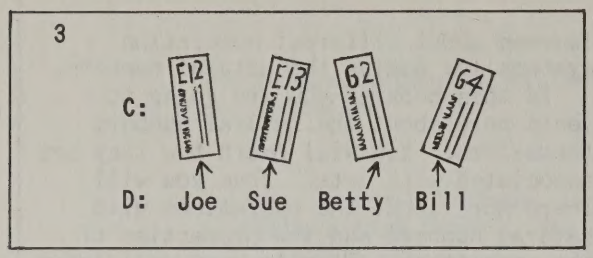
E Look again at Displays 1 and 2. Notice that in both displays we have matched, or related, tickets with students. Speaking in the language of mathematics, we say that each ticket is *mapped onto* a student. This means that a member of one set is matched with, or corresponds to, a member of another set. In Display 1, how many tickets are mapped onto Mary? How many tickets are mapped onto Steve? Is more than one ticket mapped onto a single student? The mapping in Display 1 is a *many-to-one mapping.* We say that set A is mapped onto set B.

F Look again at Display 2. How many tickets are mapped onto Joe? How many tickets are mapped onto Sue? Onto Betty? Onto Bill?

G How do you know that the mapping expressed in Display 2 is not a many-to-one mapping? The mapping expressed in Display 2 is a *one-to-one mapping.* Set C is mapped onto set D.

H In Display 2 we mapped the set of tickets that Ed sold onto the set of

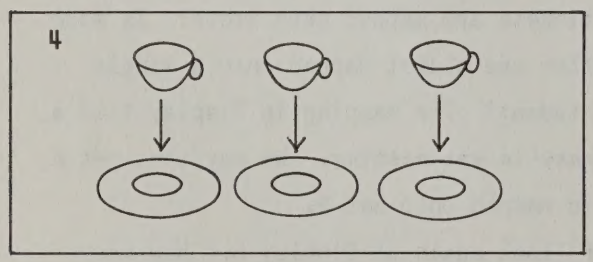
students to whom the tickets were sold. Now look at Display 3. Notice the direction in which the arrows point. We have mapped the set of students onto what set?



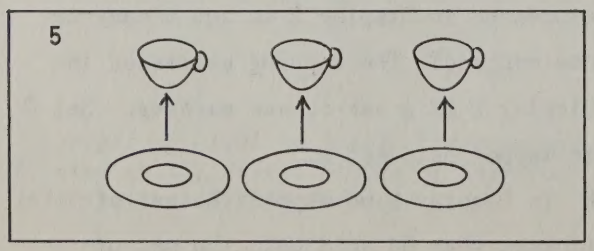
I We have mapped each member of D onto how many members of C? Is each member of C used in the mapping? Is the mapping a many-to-one mapping? Is it a one-to-one mapping? Explain your answers.

Now you can see that in a one-to-one mapping each member of one set is mapped onto just one member of a second set. Each member of the second set is used just once.

J Look at Display 4. Have we mapped each member of the set of cups onto exactly one saucer? Is each saucer used just once in the mapping?

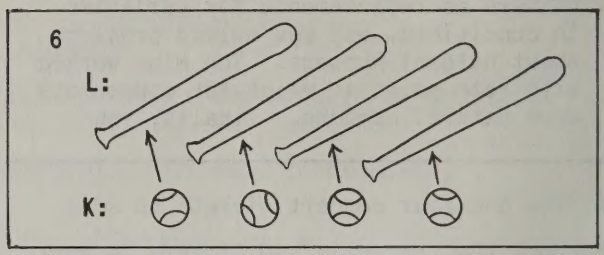


K Now look at Display 5. Have we mapped each member of the set of saucers onto exactly one cup? Is each cup used just once in the mapping?



L Are the mappings expressed in Displays 4 and 5 one-to-one mappings? How do you know?

M Look at Display 6. Explain how it expresses a one-to-one mapping of set K onto set L. Now imagine that the arrows are reversed. Is the new mapping a one-to-one mapping of set L onto set K? Explain your answer.



N Look again at the mappings expressed in Display 4 and Display 5. Whenever we have a one-to-one mapping of set A onto set B, is it always possible to have a one-to-one mapping of B onto A?

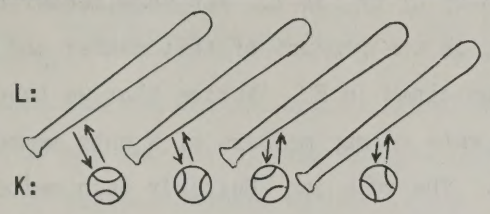
Now you know that whenever you have a one-to-one mapping of set A onto set B, you also have a one-to-one mapping of set B onto set A. To emphasize that we have *two* one-to-one mappings, we often speak of a *one-to-one correspondence* between two sets. This means that we have a one-to-one mapping of each set onto the other.

O Is the set of saucers shown in Displays 4 and 5 in one-to-one correspondence with the set of cups?

one-to-one correspondence (kôr'ə spon'dəns). Two sets, A and B, are in one-to-one correspondence when there is a one-to-one mapping of set A onto set B and there is a one-to-one mapping of set B onto set A. The set of fingers on one hand can be put in one-to-one correspondence with the set of toes on one foot.

P Now look at Display 7. We have used one arrow to show that the first bat is mapped onto the first baseball, and another arrow to show that the first baseball in turn is mapped onto the first bat. The second bat is mapped onto what member of K? The second baseball is mapped onto what member of L?

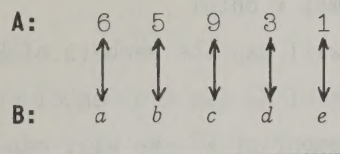
7



Q Have we put the set of bats and the set of baseballs in one-to-one correspondence?

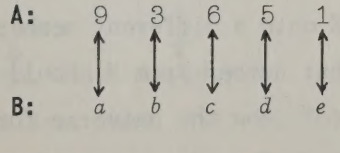
A In Display 8 we have used double-headed arrows to express the mappings. For each member of A, tell what member of B it is mapped onto. For each member of B, tell what member of A it is mapped onto. Are sets A and B in one-to-one correspondence?

8



B Now look at Display 9. Notice that the numerals for set A have been rearranged. For each member of A, tell the member of B that it is mapped onto. For each member of B, tell the member of A that it is mapped onto. Does Display 9 show another way to put sets A and B in one-to-one correspondence?

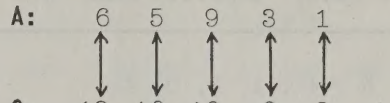
9



? **C** Can two sets be put in one-to-one correspondence in more than one way? If so, make a diagram that shows another way of putting sets A and B in one-to-one correspondence.

D Now look at Display 10. Is each member of A used in the mappings? Is each member of C used in the mappings? Have we put sets A and C in one-to-one correspondence?

10



E Make a diagram similar to the one in Display 8 to show that $\{6, 5, 9, 3, 1\}$ and $\{chair, table, apple, pear, orange\}$ can be put in one-to-one correspondence. Then show a second way to put the two sets in one-to-one correspondence.

? **F** How many possible ways are there of putting $\{1, 2, 3\}$ and $\{a, b, c\}$ in one-to-one correspondence?

G Which sets named in Display 11 are infinite sets? Explain your answer.

11

D = {Paul, Tom, Dick, Ed, Steve}.
E = {cat, dog}.
F = {0, 1, 2, 3, 4, ···}.
G = {a, b, c, d, e}.
H = {0, 2, 4, 6, 8, ···}.

H Which of the sets tabulated in Display 11 can be put in one-to-one correspondence with $\{6, 5, 9, 3, 1\}$? Use diagrams to help you obtain your answer.

I When two sets can be put in one-to-one correspondence, they are *equivalent sets*. $\{6, 5, 9, 3, 1\}$ is equivalent to which sets named in Display 11? How do you know?

equivalent sets (i kwiv' ə lənt). Two sets, A and B, are equivalent only when they can be put in one-to-one correspondence. $\{1, 2, 3\}$ and $\{a, b, c\}$ are equivalent sets.

J Is $\{6, 5, 9, 3, 1\}$ an infinite set? Can $\{6, 5, 9, 3, 1\}$ be equivalent to an infinite set? How do you know?

K Look at Display 12. Are R and S infinite sets? Describe each set.

12

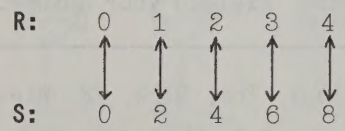
$R = \{0, 1, 2, 3, 4\}$.

$S = \{0, 2, 4, 6, 8\}$.

L Now we will see if we can put R and S in one-to-one correspondence. Multiply each member of R by 2. What products do you obtain? Is each product a member of S?

M We will map each member of R onto the member of S that is the product of 2 and the member of R. Does Display 13 express this mapping? What other mapping is expressed in Display 13?

13



N Are sets R and S in one-to-one correspondence? Are they equivalent sets?

Now you will see how infinite sets can be put in one-to-one correspondence.

A Look at Display 14. Describe each set. How do you know that each set is an infinite set?

14

$N = \{0, 1, 2, 3, 4, \cdots\}$.

$E = \{0, 2, 4, 6, 8, \cdots\}$.

B First we will map the members of N onto the members of E. Is 3 a member of N? Is 3×2 a member of E? Is 17 a member of N? Is 17×2 a member of E? Is 28 a member of N? Is 28×2 a member of E?

C We will map 3 in N onto 3×2, or 6, in E. We will map 17 in N onto 17×2, or 34, in E. We will map 28 in N onto 28×2, or 56, in E. For each member of N, is the product of that member and 2 contained in E? (Notice that we have used a *rule* to map members of N onto members of E. The rule is: Multiply each member of N by 2.) Onto what number should we map 5? 25? 500? 10,001?

D By using the rule mentioned above, is it possible to map each member of N onto a different member of E? Will each member of E be used in the mapping?

E Suppose we choose any number from N and use x as a variable for this number. The universe for x is $\{0, 1, 2, \cdots\}$. What number from E should we map x onto?

F Now we will map the members of E onto the members of N. Is 6 a member of E? Is $6 \div 2$ a member of N? We will map 6 onto $6 \div 2$, or 3.

G Is 36 a member of E? Is $36 \div 2$ a member of N? We will map 36 onto $36 \div 2$.

H Is 35 a member of E? Explain your answer.

I What rule have we used to map members of E onto members of N? Onto what members of N do we map 8? 18? 806?

J Can we use the same rule to map each member of E onto a different member of N?

K What number from N should we map x onto? Now the universe for x is $\{0, 2, 4, \cdots\}$.

Time out for a chat

You already know that set N tabulated in Display 14 is the set of natural numbers. Set E is the set of even numbers. Notice that each even number has 2 as a factor.

Sometimes 0 is not included in the set of even numbers. But 2 is a factor of 0, since $0 = 2 \times 0$ is a true statement. Therefore, we include 0 in the set of even numbers.

The odd numbers do not have 2 as a factor. The odd numbers are 1, 3, 5, and so on.

L Have we put the set of even numbers and the set of natural numbers in one-to-one correspondence? Explain your answer.

M Are the set of even numbers and the set of natural numbers equivalent sets? How do you know?

? **N** Make a diagram to show that the set of natural numbers and the set of odd numbers are equivalent sets.

? **O** Are the set of even numbers and the set of odd numbers equivalent sets? Explain your answer.

On Your Own

Notice that, in Exercises 1 through 5, two sets are tabulated in each exercise. They are named A and B. For each exercise, first tabulate a subset of set A that can be mapped by a one-to-one mapping onto set B. Then make a diagram to express the mapping of the subset of A onto B. For example, if you use the subset {*Mary, Jill, Sue*} for Exercise 1, your diagram will look like this:

{Mary, Jill, Sue}
↓ ↓ ↓
{June, July, August}

1 A = {Alice, Mary, Jill, Sue}.
B = {June, July, August}.

2 A = {0, 1, 2, 3, 4, 5, 6, 7}.
B = {Sunday, Monday, Tuesday}.

3 A = {United States, Mexico, Chile}.
B = {Canada}.

4 A = {0, 1, 2, 3, 4, 5, 6}.
B = {0, 1, 2, 3}.

5 A = {0, 1, 2, 3, 4, 5, 6}.
B = {(0,1), (1,0), (1,1)}.

Two sets are described in each of Exercises 6, 7, and 8. For each exercise, make a diagram to show that the two solution sets can be placed in one-to-one correspondence. Use a tabulation of each set in your diagram.

6a $\{x \mid x < 5\}$ $U = N$.
b $\{a \mid a - 9 < 5\}$ $U = N$.
7a $\{n \mid n/4 \sim 8/1\}$ $U = C$.
b $\{p \mid p/4 \sim 12/1\}$ $U = C$.
8a $\{(x,y) \mid x/y \sim 3/4 \wedge y < 16\}$ $U = C$.
b $\{(a,b) \mid 4/3 \sim a/b \wedge a < 15\}$ $U = C$.

For each set named in Exercises 9 through 14, find an equivalent set tabulated in Display 15. Use a diagram to explain your answer.

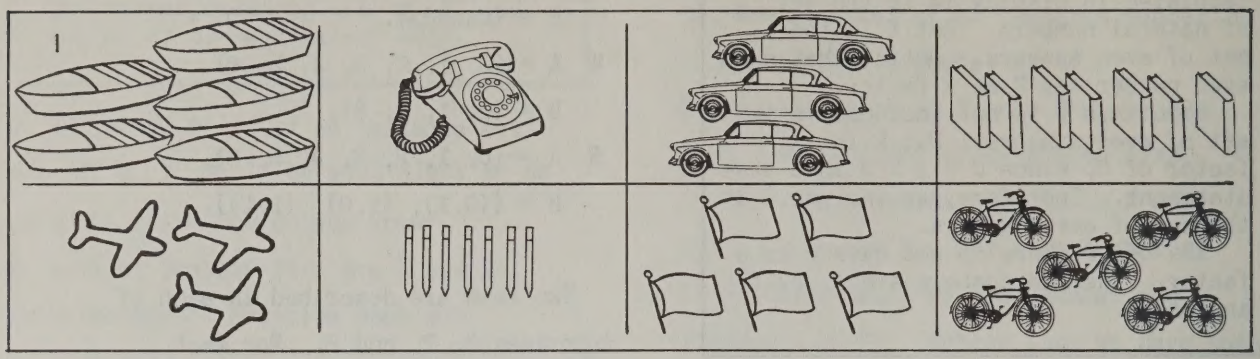
15

A = {0}.	D = {0, 1, 2, 3}.
B = {0, 1}.	E = {0, 1, 2, 3, 4}.
C = {0, 1, 2}.	F = {0, 1, 2, 3, 4, 5}.

9 {red, blue}
10 $\{x \mid x < 3\}$ $U = N$.
11 $\{x \mid 2x = 0\}$ $U = N$.
12 $\{(a,b), (c,d)\}$
13 $\{a, b, c, d\}$
14 $\{(x,y) \mid x/y \sim 5/6 \wedge x < 35\}$ $U = C$.
15 Make a diagram to show that {*10, 100, 1000, 10,000, ···*} is equivalent to the set of counting numbers.

EXPLORING IDEAS
Lesson 2
Booklet 7

In this lesson you will learn that natural numbers are properties of sets.



A Look at Display 1. Describe each set pictured in the display.

B Can the set of boats be mapped onto the set of flags? Can the set of flags be mapped onto the set of boats? Explain your answers.

C Can the set of boats and the set of flags be put in one-to-one correspondence? Is the set of boats equivalent to the set of flags?

D The set of boats can be put in one-to-one correspondence with what other set pictured in Display 1? How do you know?

E Is the set of boats equivalent to the set of bicycles? Is the set of flags equivalent to the set of bicycles?

F Do the set of boats, the set of flags, and the set of bicycles have the same number of members? Explain your answer.

G Select some other objects and tabulate a set that is equivalent to each of the sets described in Exercise F.

H Can the set of airplanes be put in one-to-one correspondence with the set of boats? How do you know?

I Which set pictured in Display 1 can be put in one-to-one correspondence with the set of airplanes? Is the set of airplanes equivalent to this set?

J Does the set of airplanes have the same number of members as the set of automobiles? Explain your answer.

K Select some other objects and tabulate another set that is equivalent to the set of airplanes.

L Does the set you tabulated in Exercise K have the same number of members as the set of airplanes? As the set of automobiles? Explain your answers.

M Is the set of pencils pictured in Display 1 equivalent to the set of automobiles? To the set of flags? To the set of books? Explain your answers.

N Does the set of pencils have the same number of members as the set of books? Explain your answer.

O Select some other objects and tabulate a set that is equivalent to each of the sets described in Exercise N.

P The set whose member is a telephone is equivalent to what other set pictured in Display 1? How do you know?

Q Is $\{Illinois\}$ equivalent to the set described in Exercise P? How do you know?

Now you see that two or more equivalent sets that are not infinite have the same *natural number*. Since these sets have the

same natural number, we say that a natural number is a *common* property of these sets. The natural number of a set that is not infinite tells how many members are in the set.

Now let us see how we can associate each of the natural numbers with a particular set and with all sets equivalent to the particular set.

A Look at Display 2. The set tabulated there has how many members? What is a name for the set that has no members?

> *Reminder*: When a set has no members, it is called the empty set. The set of women who have been President of the United States is one description of the empty set. (See page 21 of BOOKLET 1 or page 10 of BOOKLET 6.)

2

$\{\ \}$

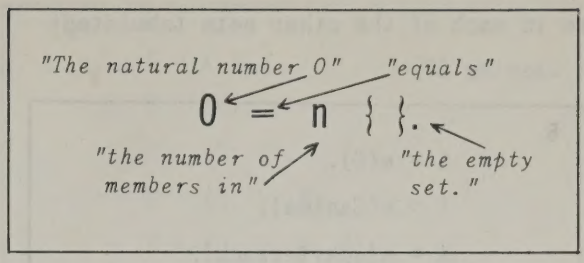
B Give another description of the empty set.

C Is the set of women who have been President of the United States the same set as the set of students in your class who are under five years of age? Why is each of the sets described the empty set?

Since the empty set is the set that has no members, the *natural number 0* is associated with the empty set. We will use the sentence at the right to show that the natural number 0 is the number of members in the empty set. The diagram at the top of the next column shows how to read this sentence.

$0 = n\{\ \}.$

You have learned that you associate the empty set with the natural number 0. Now



we will choose a particular set to associate with the number 1. Since numbers are mathematical objects, we will choose a set whose member is a number. So far 0 is the only number we have associated with a set. Let us see how we can use the set whose member is 0 to help us understand the meaning of the number 1.

D Look at Display 3. Describe the set.

3

$\{0\}$

E How does $\{\ \}$ differ from $\{0\}$? How does $n\{\ \}$ differ from $\{0\}$? Explain your answers.

F Look at Display 4. Describe each set.

4

A = {Canada}.
B = {quarterback}.
C = {*a*}.

G Is each set that is tabulated in Display 4 equivalent to each of the other sets in the display? Is each set that is tabulated in Display 4 equivalent to the set tabulated in Display 3? Explain your answers.

H Tabulate a set that is equivalent to each of the sets that are tabulated in Displays 3 and 4.

I Read the sentences in Display 5 on the next page. The natural number 1 is the number of members in $\{0\}$. How many members

are in each of the other sets tabulated in Display 5?

5

$1 = n\{0\}$.

$1 = n\{\text{Canada}\}$.

$1 = n\{\text{quarterback}\}$.

$1 = n\{a\}$.

J What is the natural number of the set that you tabulated in Exercise H? How do you know?

K What is a common mathematical property of $\{0\}$ and each set equivalent to $\{0\}$?

Now we will use the natural numbers 0 and 1 as objects to obtain a set associated with another natural number.

L Look at Display 6. Describe the set.

6

$\{0, 1\}$

M Look at Display 7. Describe each set. Is each set that is tabulated here equivalent to each of the other sets in the display? How do you know?

7

A = {France, England}.

B = $\{a, b\}$.

C = {hat, coat}.

N Is each set tabulated in Display 7 equivalent to the set tabulated in Display 6? How do you know?

O Tabulate a set that is equivalent to each of the sets that are tabulated in Displays 6 and 7.

P Read the sentence in Display 8. The natural number 2 is the number of members in $\{0, 1\}$. Is 2 also the number of members in each set tabulated in Display 7?

8

$2 = n\{0, 1\}$.

Q What is a common mathematical property of $\{0, 1\}$ and each set equivalent to $\{0, 1\}$?

R In Display 9, what is the natural number of each set tabulated?

9

A = { }.

B = {0}.

C = {0, 1}.

Each of the sets tabulated in Display 9 is a *standard* set. Notice that a standard set either is empty or has natural numbers for its members. When you think about a natural number, you need think only of the standard set with which it is associated. For example, you should remember that 2 is the natural number of $\{0, 1\}$. Then, if you are working with $\{Minnesota, Michigan\}$, you know that 2 is also the natural number of $\{Minnesota, Michigan\}$, since $\{Minnesota, Michigan\}$ is equivalent to $\{0, 1\}$. 2 is also the natural number of any other set equivalent to $\{0, 1\}$.

Now we will see how other natural numbers are associated with a standard set and with each set equivalent to the standard set.

A In Display 10, describe the set tabulated.

10

$\{0, 1, 2\}$

B Look at Display 11. Describe each set. Is A equivalent to B? How do you know?

11

A = {Ohio, Iowa, Texas}.

B = {trumpet, drum, clarinet}.

C Is each set tabulated in Display 11 equivalent to the set tabulated in Display 10? How do you know?

D Read the sentence in Display 12. The natural number 3 is the number of members in $\{0, 1, 2\}$. Is $\{0, 1, 2\}$ the standard set of 3? How do you know?

12

$3 = n\{0, 1, 2\}.$

E What natural number is the number of members in each set tabulated in Display 11? What is a common property of $\{0, 1, 2\}$ and the sets tabulated in Display 11?

F What is a common property of $\{0, 1, 2\}$ and each set equivalent to $\{0, 1, 2\}$?

G Read the sentence in Display 13. The natural number 4 is the number of members in $\{0, 1, 2, 3\}$. Is $\{0, 1, 2, 3\}$ the standard set of the natural number 4? How do you know?

13

$4 = n\{0, 1, 2, 3\}.$

H Tabulate two sets that are equivalent to the standard set of the natural number 4. What is a common property of the sets that you tabulated?

I What is a common property of $\{0, 1, 2, 3\}$ and the sets that you tabulated in Exercise H?

J What is a common property of $\{0, 1, 2, 3\}$ and each set equivalent to $\{0, 1, 2, 3\}$?

Now you have seen that a natural number is a common property of a standard set and all sets equivalent to the standard set.

K Look at Display 14. For each set tabulated, tell what natural number is associated with the standard set.

14

$A = \{\ \}.$

$B = \{0\}.$

$C = \{0, 1\}.$

$D = \{0, 1, 2\}.$

$E = \{0, 1, 2, 3\}.$

$F = \{0, 1, 2, 3, 4\}.$

$G = \{0, 1, 2, 3, 4, 5\}.$

Notice that the standard set of a natural number is made up of just those numbers already developed. For example, the standard set of 5 has for its members the numbers 0, 1, 2, 3, and 4.

L Tabulate the set whose members are the days of the week.

M Look at Display 15. Is the set tabulated here equivalent to the set you tabulated in Exercise L? How do you know?

15

$\{0, 1, 2, 3, 4, 5, 6\}$

N What is a common property of the set you tabulated in Exercise L and the set tabulated in Display 15?

O Look at Display 16. The set tabulated is the standard set of what natural number? How do you know?

16

$\{0, 1, 2, 3, 4, 5,$
$6, 7, 8, 9, 10, 11\}$

P Look again at Display 14. Tabulate the standard set of 8. Of 9. Of 19. Of 20.

? **Q** Suppose m is a variable that can be replaced by a natural number. Tabulate the standard set of m. Describe in words how you could form the standard set for any given natural number.

R Now we will collect the natural numbers. Look at Display 17. We can describe the set tabulated here as *the set of natural numbers*.

17 $N = \{0, 1, 2, 3, \cdots\}$.

S What letter is used to indicate the set of natural numbers? Is the set of natural numbers an infinite set?

? **T** Is the standard set of any natural number an infinite set? Explain your answer.

On Your Own

A set is described in each of Exercises 1 through 5. For each exercise, tabulate a subset that is equivalent to $\{0, 1, 2, 3, 4\}$.

1 set of rivers

2 set of vegetables

3 set of furniture

4 set of birds

5 set of games

6 What is a common property of the sets that you tabulated in Exercises 1 through 5?

A natural number is expressed in each of Exercises 7 through 12. For each exercise, write a sentence that expresses the standard set of the natural number. For Exercise 7 you should write the following:

$13 = n\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

7 13 **8** 10 **9** 0

10 11 **11** 15 **12** 21

Tabulate the standard set equivalent to each set tabulated in Exercises 13 through 22.

13 {apple, pear, watermelon, peach}

14 {elm, willow, oak, maple, sycamore, walnut}

15 {sparrow, robin}

16 {bicycle}

17 {Washington, Jefferson, Lincoln}

18 {taxi, bus, train, airplane, ship}

19 $\{\neq, \times, +, =, \wedge, >, <\}$

20 { }

21 {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27}

22 $\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o\}$

In each of Exercises 23 through 32, the standard set of a natural number is tabulated. For each exercise write a sentence that expresses the natural number of the standard set. For exercise 23 you should write the following:

$11 = n\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

23 {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

24 {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13}

25 {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17}

26 {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}

27 {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14}

28 {0, 1, 2, 3, 4, 5, 6, 7, 8}

29 {0}

30 {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16}

31 {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

32 $\{0, 1, 2, 3, \cdots, m-1\}$

33 Is the standard set of the natural number 5 equivalent to the standard set of the natural number 6? Explain your answer.

In this lesson you will learn about some of the properties of natural numbers.

Time out for a chat

In Lessons 1 and 2 you studied equivalent sets. You learned that every natural number is associated with a standard set and with each set equivalent to the standard set.

Natural numbers have many characteristics, or, in the language of mathematics, they have many properties. In this lesson you will learn about some of these properties.

You have often used the ideas of "is less than" and "is greater than." Now that you have studied about equivalent sets, it is possible to state exactly what these ideas mean. But first you must learn more about subsets.

Reminder: If each member of one set is also a member of a second set, then the first set is a subset of the second set. (See page 7 of BOOKLET 1 or page 3 of BOOKLET 6.)

A Look at the sets tabulated in Display 1. Is each member of {*cat, dog, lion*} a member of S? How do you know that {*cat, dog, lion*} is a subset of S?

1

S = {cat, dog, lion, tiger}.

{cat, dog, lion} {cat, dog}
{cat, lion} {tiger} {tiger, dog}

B Does {*cat, dog, lion*} contain all the members of S?

You know that {*cat, dog, lion*} is a subset of S. You also know that {*cat, dog, lion*} does not contain all the members of S. We call {*cat, dog, lion*} a *proper* subset of S. When a subset does not contain all the members of a set, that subset is a proper subset.

C Is {*cat, dog*} a subset of S? Is {*cat, dog*} a proper subset? How do you know?

D Which of the other sets tabulated in Display 1 are proper subsets of S? Explain your answer.

E Is each member of {*dog, cat, tiger, lion*} a member of S? Does {*dog, cat, tiger, lion*} contain each of the members of S? Is S a subset of itself? Is S a proper subset of itself? How do you know?

F Is {*0*} a proper subset of {*0, 1*}? How do you know?

G Tabulate four proper subsets of the standard set of 3.

Now we will use the ideas of proper subset and standard set to learn about the ideas of "is less than" and "is greater than."

H Look at Display 2. Is the standard set of 1 a proper subset of the standard set of 2? How do you know?

2

$1 = n\{0\}$.	$2 = n\{0, 1\}$.
$2 = n\{0, 1\}$.	$3 = n\{0, 1, 2\}$.
$1 < 2$.	$2 < 3$.
$2 > 1$.	$3 > 2$.

I Is the standard set of 2 a proper subset of the standard set of 3? How do you know?

J We will agree that a number is less than another number only when the standard set of the first number is a proper subset of the standard set of the second number. How do you know that $1 < 2$ is a true statement? How do you know that $2 < 3$ is a true statement?

K The inequality $2 > 1$ is the same relationship between the numbers 2 and 1 as is the inequality $1 < 2$. We say that $2 > 1$ gives us the same information as $1 < 2$. In other words, $2 > 1$ means $1 < 2$. Does the inequality $3 > 2$ give us the same information as $2 < 3$?

L Tabulate the standard set of 4. Is the standard set of 3 a proper subset of the standard set of 4? How do you know?

M Which number is less, the number 3 or the number 4? How does your answer to Exercise L help you answer the question?

N Is $3 < 4$ a true statement? What other inequality gives us the same information as $3 < 4$?

O So far we have shown that $1 < 2$, $2 < 3$, and $3 < 4$ are true statements. Now let us see why $4 < 5$ is also a true statement. Tabulate the standard set of 5. Now explain why $4 < 5$ is a true statement.

P What other inequality gives us the same information as $4 < 5$?

Q Is $5 < 6$ a true statement? Explain your answer.

R What other inequality gives us the same information as $5 < 6$?

We can continue in this way to show that $6 < 7$, $7 < 8$, $8 < 9$, and so on, are true statements. Now let us use variables to make conditions for inequalities that refer to the natural numbers.

A Two conditions are expressed in Display 3. Use 7 as a replacement for a, and 8 as a replacement for b in these conditions. From which condition do you obtain a true statement?

3

Universe for a and b = $\{1, 2, 3, \cdots\}$.

The standard set of a is a proper subset of the standard set of b.

The standard set of b is a proper subset of the standard set of a.

B Now look at Display 4. Use 7 as a replacement for a and 8 as a replacement for b. From which condition will you obtain a true statement?

4

$a < b$. $\qquad$ $b < a$.

C Look again at Display 3. Use 10 for a and 9 for b. From which condition do you obtain a true statement?

D Using 10 for a and 9 for b, from which condition expressed in Display 4 do you obtain a true statement?

? **E** For any members of $\{1, 2, 3, \cdots\}$ that are replacements for a and b, when will you obtain a true statement from $a < b$?

Time out for a chat

Now you know what is meant by $a < b$ for each replacement of a and b from $\{1, 2, 3, \cdots\}$. So far, however, we have not included the number zero in our universe. We will agree that zero is less than each of the other natural numbers. Later, as you progress in your study of mathematics, you will see how the empty set can be used in deciding that zero is less than each of the other natural numbers.

Now you will learn how the idea of "is less than" can help you think of the natural numbers in a special order.

F Are $0 < 1$, $0 < 2$, $0 < 3$, $0 < 4$, and $0 < 5$ true statements?

G Tabulate $\{x \mid 0 < x\}$. $U = N$.

H Are $1 < 2$, $1 < 3$, $1 < 4$, and $1 < 5$ true statements? Use standard sets to explain your answer.

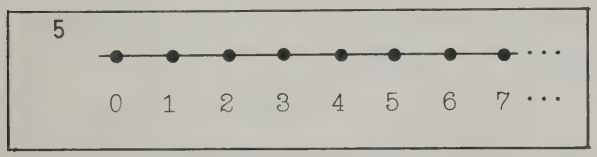
I Tabulate $\{x \mid 1 < x\}$. $U = N$.

J Is the number 1 less than each of the other natural numbers except 0?

K Tabulate the solution set of $2 < x$. $U = N$. Is 2 less than each of the other natural numbers except 0 and 1? How do you know?

L We can continue in this way to order the members of the set of natural numbers. 0 is first, since 0 is the least natural number; then come 1, 2, 3, and so on. Will each number be less than each of the numbers after it?

M Now look at the graph of the set of natural numbers in Display 5. The point farthest to the left is assigned to what number? Is this the least natural number?



N Is point 1 immediately to the right of point 0? Is 1 less than each of the other natural numbers except 0?

O Is point 2 immediately to the right of point 1? Is 2 less than each of the other natural numbers except 0 and 1?

P Why do we locate point 3 immediately to the right of point 2?

Q The number 4 is assigned to the point that is four units to the right of point 0. What number is assigned to the point that is seven units to the right of point 0? Ten units to the right of point 0? Is 0 less than each number?

R Is the position of point 0 to the left of the positions of the points assigned to numbers greater than 0?

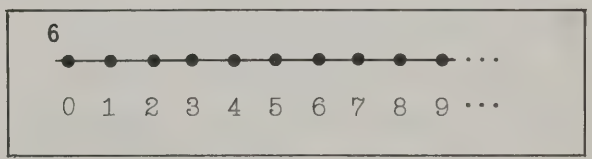
S How is the position of point 1 related to the positions of the points assigned to numbers greater than 1?

T How is the position of point 2 related to the positions of the points assigned to numbers greater than 2?

U If we obtain a true statement from the condition $a < b$, how is the position of point a related to the position of point b on the number line? $U = N$.

Since the natural numbers can be arranged in a special order by using the idea of "is less than," we say that the natural numbers have the property of *order*. Each number in the order 0, 1, 2, and so on, is less than each of the numbers that follow. A graph helps us think about this order. Now we will use a graph to help us discover two more properties of the natural numbers.

A Look at Display 6. What number comes next after 7? Next after 37? Next after 0? Next after 5000?



B Does each natural number have a natural number that immediately follows it? If so, how could you find this number for any give natural number?

C The number that immediately follows a given number is the *successor* of the

given number. What is the successor of 5? Of 0? Of 73,925? Does each natural number have a successor?

? **D** Is it possible for two different natural numbers to have the same successor? Explain your answer.

? **E** Is there a natural number whose successor is 0? Explain your answer.

? **F** Suppose a set of natural numbers contains 0 and the successor of each of its members. Will this set contain all the natural numbers?

Since each natural number has a successor, we say that the natural numbers have the *successor* property.

G Now look again at Display 6 on the preceding page. Let us think of points between two given points on the number line. What points are between point 5 and point 10? Does each of the numbers assigned to these points satisfy the compound condition $5 < x \wedge x < 10$? Explain your answer. Tabulate $\{x \mid 5 < x \wedge x < 10\}$. The universe is N.

H Now tabulate the set of numbers whose points are between point 8 and point 16. Is this set an infinite set? This set is the solution set of what compound condition?

? **I** Think of any two points on the number line; then tabulate the set of natural numbers whose points are between the points you chose. Is this set an infinite set? What compound condition is satisfied by the numbers whose points are between the points you chose?

? **J** Is there a natural number that is greater than 7 and less than 8? Explain your answer.

? **K** Suppose that the natural numbers are arranged in order according to the idea of "is less than." Tabulate the set of numbers between 10 and 11.

You have seen that the natural numbers have the successor property. You have also seen that it is possible to determine how many natural numbers are between any two natural numbers. It is said that natural numbers have the property of *betweenness*.

Later on you will study numbers that are not natural numbers. Then you will notice that each of these numbers will *not* have a successor, and you will *not* be able to determine how many numbers are between any two of them. Not all kinds of numbers have the same properties.

On Your Own

1 Tabulate three proper subsets of each of the following sets. $U = N$.

a $\{0, 1, 2\}$

b $\{x \mid x + 3 < 9\}$

c $\{(1,2), (1,3), (1,4)\}$

d $\{n \mid 5n < 50\}$

e $\{a, b, c, d\}$

f $\{a \mid 9 - a > 3\}$

For each ordered pair named in Exercises 2 through 7, tell whether the standard set associated with the first component is a proper subset of the standard set associated with the second component.

2 (6,9) **4** (15,10) **6** (2,11)

3 (1,7) **5** (11,9) **7** (10,4)

8 Which of the ordered pairs named in Exercises 2 through 7 are members of $\{(x,y) \mid x < y\}$? $U = N \times N$.

9 Is $\{(x,y) \mid x < y\}$ the same set as $\{(x,y) \mid y > x\}$? How do you know? $U = N \times N$.

10 What is the successor of the successor of 7?

11 Suppose the natural numbers are arranged in order. Tabulate the set of numbers between 9 and 21.

12 Tabulate $\{x \mid 0 < x \wedge x < 14\}$. $U = N$.

13 Tabulate $\{x \mid 35 < x \wedge x < 36\}$. $U = N$.

EXPLORING IDEAS
Lesson 4
Booklet 7

In this lesson you will learn about the sum of two natural numbers and the difference of two natural numbers.

In the first three lessons of this booklet you learned that natural numbers are associated with sets, and you ordered the natural numbers.

The set of natural numbers is a part of the natural number system. In this booklet you will learn what is meant by the natural number system. To prepare for this, you will first learn about sums and products of natural numbers. Then you will learn about addition and multiplication of natural numbers. Finally, you will learn about the properties of addition and multiplication that are a part of the natural number system.

A Three sets are tabulated in Display 1. Joe is a member of set T. Is Joe a member of set R? Of set S?

1

R = {Joe, Ed, Sally}.
S = {Sally, Ann, Bill}.
T = {Joe, Ed, Sally, Ann, Bill}.

B For each member of T, tell whether that member is also in R, in S, or in both R and S.

C How have we used set R and set S to obtain set T? We call set T the *union* of R and S.

union (un'yən). The union of set A and set B is the set of all members of A together with all members of B. It does *not* include any other objects. If $A = \{0, 1, 2, 3\}$ and $B = \{2, 3, 4\}$, then the union of A and B is $\{0, 1, 2, 3, 4\}$.

D Now look at Display 2. Will Hong Kong be a member of the union of K and L? Will Rome be a member of the union of K and L? Explain your answers.

2

K = {Berlin, Paris, London}.
L = {Hong Kong, Tokyo}.

E Is {*Berlin, Paris, London, Hong Kong, Tokyo*} the union of K and L? How do you know?

F How do you know that {*London, Hong Kong, Tokyo*} is not the union of K and L?

G The symbol at the right is read "the union of K and L." This symbol is a name of the set that is the union of K and L. Now read the sentence in Display 3 at the top of the next page.

$K \cup L$

H Does the sentence in Display 3 express a true statement? How do you know?

I Now look at Display 4 on the next page. Tabulate each set named. $U = N$.

3

$K \cup L = \{\text{Berlin, Paris, London, Hong Kong, Tokyo}\}.$

"The union of K and L" — *"is equal to"* — *"the set whose members are"* — *"Berlin, Paris, London, Hong Kong, Tokyo."*

Reminder: The members of set B named in Display 4 satisfy a compound condition. The compound condition is made up of the simple condition $2x < 30$ and the simple condition $x > 11$. The symbol for "and" ($\wedge$) means that each member of set B must satisfy both simple conditions in the compound condition. (See pages 49 and 71 of BOOKLET 3 or pages 28 and 29 of BOOKLET 6.)

4

$A = \{x \mid x < 3\}.$

$B = \{x \mid 2x < 30 \wedge x > 11\}.$

J Will 11 be a member of the union of A and B? Will 2 be a member of the union? Explain your answers.

K Tabulate the union of A and B. Is the symbol $A \cup B$ a name for the set that you tabulate?

L Now look at Display 5. Is $F = D \cup E$ a true statement? How do you know?

5

D = {Mexico, Puerto Rico, Spain}.

E = {France, Spain}.

F = {Mexico, Puerto Rico, Spain, France}.

M $A = \{2, 4, 8, 16\}$. Tabulate $A \cup A$. Remember that in tabulating a set we list the name of each member just once.

N Look again at Displays 1 and 2. Are the members of R entirely different members from the members of S? Are the members of K entirely different members from the members of L?

If two sets have entirely different members, the two sets have *no common members*. When two sets have no common members, no object is a member of both sets.

O Look again at Display 4. Do set A and set B have common members? Two sets like A and B that have no common members are *disjoint sets*.

P Which of the sets tabulated in Display 1 are disjoint sets? Are the sets tabulated in Display 2 disjoint sets? How do you know? Which of the sets tabulated in Display 5 are disjoint sets? Explain your answer.

disjoint sets (dis joint'). Two sets are disjoint sets only when the sets have no common members. $\{0, 1, 2\}$ and $\{3, 4, 5\}$ are disjoint sets. $\{0, 1, 2\}$ and $\{2, 3, 4\}$ are not disjoint sets.

Q Are $\{0, 1\}$ and $\{2, 4\}$ disjoint sets? Tabulate the union of these sets. What number is associated with the union of $\{0, 1\}$ and $\{2, 4\}$?

R Are $\{0, 1\}$ and $\{0, 2\}$ disjoint sets? Tabulate the union of these sets. What number is associated with the union?

Now let us see how the natural numbers of two disjoint sets are related to the natural number of the union of the sets.

A Are the sets tabulated in Display 6 disjoint sets? For each set tell what number is associated with the set.

6

A = {1, 3}. B = {0, 2, 4}.

B The symbol at the right is read "the number of members in set A." Look again at set A tabulated in Display 6. Are $n(A)$ and $n\{1, 3\}$ the same number? How do you know?

$n(A)$

C Now read the sentence in Display 7. Does the sentence express a true statement about set A named in Display 6?

7

$2 = n(A).$

"Two" "is equal to" "the number of members in A."

D The symbol at the right is read "the number of members in the union of A and B." Tabulate $A \cup B$ for the sets named in Display 6.

$n(A \cup B)$

E You know that the number 2 is associated with set A and that the number 3 is associated with set B. Which sentences in Display 8 express these statements?

8

A $2 = n(A)$.

B $3 = n(B)$.

C $2 + 3 = n(A \cup B)$.

You also know that A and B are disjoint sets. The natural number associated with the union of A and B is the *sum* of the number 2 and the number 3. The sum is the number of members in the union of A and B. The number $2+3$ is the sum of 2 and 3.

F Now read sentence C in Display 8. Is $2 + 3 = n(A \cup B)$ a true statement?

G Look at Display 9. How do you know that E and F are disjoint sets?

9

E = {Kentucky}.

F = {Florida, Georgia, Alabama, Louisiana, Texas}.

H Tabulate $E \cup F$.

I Read each sentence in Display 10. Does each sentence express a true statement for the sets named in Display 9? The sum of 1 and 5 is the number associated with what set?

10

A $1 = n(E)$.

B $5 = n(F)$.

C $1 + 5 = n(E \cup F)$.

J Look again at Display 6 and Display 9. Two disjoint sets are named in each display. Is the sum of the number associated with A and the number associated with B equal to the number associated with the union of A and B? How are the numbers associated with E and F related to the number associated with the union of E and F?

You have learned that the number associated with the union of two disjoint sets is the sum of the numbers associated with the disjoint sets. Now let us use variables to help us make a definition of *sum*. We will begin by letting S and T represent any two disjoint sets that are associated with natural numbers.

K If 8 is associated with S, and 4 is associated with T, then $3+4$ is the number associated with what set?

L Now suppose that 5 is associated with S, and 9 is associated with T. The sum $5+9$ is associated with what set?

M Now let us agree to associate s with set S and t with set T. The universe for s and t is N. After replacement, the sum $s+t$ will be the number associated with what set?

sum of two natural numbers (sum). The universe for s and t is N. Suppose that s is associated with set S, and t is associated with set T. If S and T are disjoint sets, the sum $s+t$ is the number associated with $S \cup T$. If $S = \{15, 16, 17\}$ and $T = \{51, 52\}$, then the sum of 3 and 2, $3+2$, is the number associated with $\{15, 16, 17, 51, 52\}$.

Notice that in the definition of *sum* we say that the sum $s+t$ is the number associated with $S \cup T$. Of course, we mean the sum we obtain from $s+t$ by replacing s and t by natural numbers.

When we use variables we often speak of them as if they were numbers. But we must always remember that we are talking about the numbers that we obtain *after* replacements. It is for this reason that we must always specify our universe.

Variables make it easy for us to think about such ideas as definitions and properties without using specific examples.

N Tabulate two disjoint sets, G and H, so that $5 = n(G)$ and $9 = n(H)$. Tabulate $G \cup H$.

O The sum $5+9$ is the number associated with what set that you tabulated in Exercise N? Is $5 + 9 = n(G \cup H)$ a true statement?

P Suppose that $A = \{16, 17, 18, 19\}$ and $B = \{18, 19, 20, 21, 22\}$. Is $4+5$ the number associated with $A \cup B$? Explain your answer.

Time out for a chat

In this lesson you have learned what a sum is. You used the symbol for "plus" to express a sum. When you write the symbol $3+4$, you indicate you are thinking of the number $3+4$ as a sum of two natural numbers.

When you write the symbol $3+4$, you have already added the number 3 and the number 4. When you write the sentence $3 + 4 = 7$, you are expressing the sum of 3 and 4 in two different ways. Now let us use the idea of equivalent sets to show that $3+4$ and 7 are the same number.

You know that $3 = n\{a, b, c\}$ and $4 = n\{d, e, f, g\}$ are true statements. From the definition of *sum* you know that $3+4$ is the number associated with $\{a, b, c, d, e, f, g\}$. $\{a, b, c, d, e, f, g\}$ is equivalent to the standard set $\{0, 1, 2, 3, 4, 5, 6\}$. The number associated with this standard set is 7; therefore, the number associated with $\{a, b, c, d, e, f, g\}$ is also 7. Since just one number is associated with a set, 7 and $3+4$ must be the same number. How do you know that $2+4$ and 6 are the same number?

Now you will learn something new about addition. You will see that addition is a special kind of mapping between two sets.

A On Monday Jack answered nine questions correctly on a history test. On Friday

he answered eight questions correctly on a history test. We can think of Jack's scores as the ordered pair named at the right. What does the first component refer to? What does the second component refer to?

(9,8)

Reminder: An ordered pair is a pair of objects that occur in a special order. The object that occurs first is called the first component. The object that occurs second is called the second component. (See page 3 of BOOKLET 3 or page 35 of BOOKLET 6.)

B Now look at Display 11. The chart shows the names of Jack and some of his classmates and expresses the scores they made on the two history tests.

11

	Jack	Nancy	Marie	Roger
Test 1	9	7	9	10
Test 2	8	9	10	9

C We have used $(9,8)$ to represent Jack's scores. What ordered pair represents Nancy's scores? Marie's scores? Roger's scores?

D Now look at Display 12. Is S the set of ordered pairs representing the scores made by Jack, Nancy, Marie, and Roger?

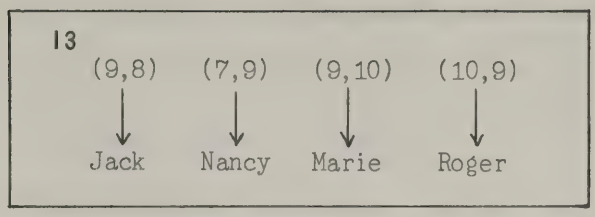
12

$S = \{(9,8), (7,9), (9,10), (10,9)\}$.

E Now look at Display 13 in the next column. The first arrow shows that we have mapped $(9,8)$ onto Jack. Have we mapped Jack's scores onto Jack?

F Have we mapped Nancy's scores onto Nancy? Marie's scores onto Marie? Roger's scores onto Roger?

13

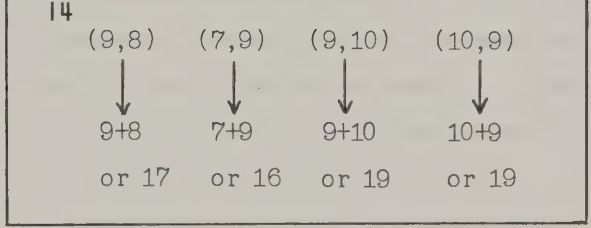


G Have we mapped the set of ordered pairs of scores onto the set of students? Is the mapping a one-to-one mapping? How do you know?

Notice that the mapping expressed in Display 13 is different from the mappings you have studied so far. Here we have used a set of ordered pairs as one of the sets in the mapping. We still consider this a one-to-one mapping because we are thinking of each ordered pair as one object.

H Look at Display 14. What ordered pair of natural numbers has been mapped onto the natural number 17? The sum of the components of this pair is what number?

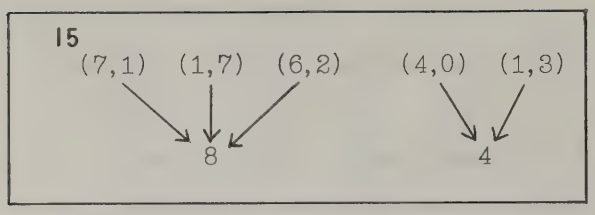
14



I Have we mapped $(7,9)$ onto the sum of 7 and 9? What was done to map each of the other ordered pairs of natural numbers onto a natural number?

J Now map each of the following ordered pairs onto the sum of its components. Onto what natural number should you map $(4,3)$? $(3,4)$? $(6,1)$? $(1,6)$?

K Look at Display 15 on the next page. Have we mapped each ordered pair named in the display onto the natural number that is the sum of the components of the pair?



L Why do we map each of the first three ordered pairs onto 8? What other ordered pairs could be mapped onto 8?

M Why do we map both $(4,0)$ and $(1,3)$ onto 4? What other ordered pairs of natural numbers could be mapped onto 4?

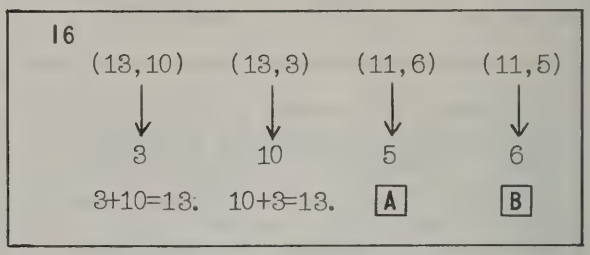
N Look again at Displays 14 and 15. Is the mapping expressed in the two displays a one-to-one mapping? Explain your answer.

O When an ordered pair is mapped onto the sum of its components, the mapping is the *operation of addition*. We can think of the operation of addition of natural numbers as a special many-to-one mapping of the set of ordered pairs of natural numbers onto another set. What is this other set?

P The universe for the variables is N. Using the operation of addition, we map (a,b) onto c only when the sum of the replacements for a and b is what number?

You have seen that any two natural numbers can be used to obtain a third natural number. This third number is the sum of the two numbers. If we begin with $(3,5)$, we obtain the number $3+5$, or 8. If we begin with $(9,20)$, we obtain the number $9+20$, or 29. We begin with two natural numbers, not with just a single number. We say, therefore, that the operation of addition is a *binary* operation. This means that addition is an operation on *two* natural numbers. Now you will learn about another binary operation.

A Look at Display 16. The mapping shown here is different from the mappings shown in Displays 14 and 15. We have mapped $(13,10)$ onto 3. Is the sum of 3 and the second component of $(13,10)$ equal to the first component of $(13,10)$?



B $(13,3)$ is mapped onto what number? Is the sum of this number and the second component of $(13,3)$ equal to the first component of $(13,3)$?

C Onto what number is $(11,6)$ mapped? Onto what number is $(11,5)$ mapped?

D How did we use addition to map each of the ordered pairs named in Display 16 onto a natural number? What sentence should we use to replace "box" A? "Box" B?

E You know that $3 + 5 = 8$ is a true statement. How does this help you find the number that $(8,5)$ is mapped onto?

The number 3 onto which we map $(8,5)$ is the *difference* of 8 and 5. The difference of 8 and 5 is $8-5$, or 3. Notice that the difference of 8 and 5 is the result of subtracting 5 from 8. When we talk about the difference of two numbers, we mean the result of subtracting the second from the first.

F You know that $6 + 2 = 8$ is a true statement. How does this help you find the number that $(8,2)$ is mapped onto?

G Is 6, or $8-2$, the difference of 8 and 2? How do you know?

H Since the sum of 2 and 7 is 9, the difference of 9 and 7 is what number? The difference of 9 and 2 is what number? How do you know?

I Look again at Display 16. Have we mapped each ordered pair of natural numbers onto a natural number that is the difference of the first and second components of the pair?

J Now map each of the following ordered pairs onto the difference of its components. Onto what number should you map $(10,2)$? $(10,8)$? $(14,3)$? $(20,20)$?

When an ordered pair is mapped onto the difference of its first component and its second component, the mapping is the *operation of subtraction*. If we begin with $(9,3)$, we obtain the difference $9-3$, or 6. If we begin with $(7,2)$, we obtain the difference $7-2$, or 5.

K The universe for the variables is N. Using the operation of subtraction, we map (a,b) onto c only when the difference of the replacements for a and b is what number?

L Is the operation of subtraction an operation on two natural numbers? Is the operation of subtraction a binary operation? How do you know?

M Is the operation of subtraction a many-to-one mapping?

You notice that we have thought of subtraction as it is related to addition. $10 - 3 = x$ means to find the replacement for x that will enable you to obtain a true statement from $10 = x + 3$. Now let us study more about addition and subtraction of natural numbers.

N In addition, we map $(5,1)$ onto 6. In subtraction, we map $(5,1)$ onto what number?

O Look at Display 17. For each ordered pair named, tell what number the pair is mapped onto by addition.

17

(5,1) (1,5) (9,3) (3,9)

P For each ordered pair of natural numbers, is there always a natural number that is the sum of the components of the pair?

Q Now we will use subtraction to map the ordered pairs named in Display 17. In subtraction, is $(5,1)$ mapped onto a number so that the sum of 1 and that number is 5?

R How do you know that $5-1$, or 4, is the difference of 5 and 1?

S In subtraction, $(1,5)$ should be mapped onto a number so that the sum of that number and 5 is what number?

T How do you know that the difference of 1 and 5 is not a natural number?

U In subtraction, is $(5,2)$ mapped onto a natural number? How do you know?

V In subtraction, is $(2,5)$ mapped onto a natural number? How do you know?

W Name three ordered pairs such that the first component of each pair is greater than the second component of that pair. For each pair you named, tell what number the pair is mapped onto by subtraction.

X Now name three ordered pairs such that the first component of each pair is less than the second component of that pair. Is each pair you named mapped onto a natural number by subtraction? Explain your answer.

Y For each ordered pair of natural numbers, is there always a natural number that is the difference of the components of the pair? Explain your answer.

? **Z** The universe for the variables is N. The difference of a and b will *not* be a natural number only if the replacement for a is less than what number?

You have learned about the set of natural numbers and the operations of addition and subtraction. You have seen that the sum of any two members of N is also a member of N. To express this in a shorter way, we say that N is *closed under addition.* N is closed under addition because the sum of any two natural numbers is always a natural number.

You also observed that the difference of two natural numbers is *not* always a natural number. N is *not* closed under the operation of subtraction.

A Now look at Display 18. Describe each set tabulated in the display.

18

$A = \{1, 2, 3, 4, 5, 6\}$.

$B = \{1, 3, 5, 7, \cdots\}$.

B Is each set named in Display 18 a subset of N? How do you know?

C 2, 3, and 4 are members of A. Is $2+3$ a member of A? Is $3+4$ a member of A?

D Is the sum of two members of A always a member of A? Explain your answer.

E Is set A closed under the operation of addition? How do you know?

F Is $3-3$ a member of A?

G Is set A closed under subtraction? Explain your answer.

H Now look at set B named in Display 18. Are 3, 5, and 9 members of B? Is $3+5$ in B? Is $5+9$ in B?

I Is set B closed under addition?

? **J** Is B closed under subtraction? Explain your answer.

? **K** $C = \{0, 2, 4, 6, \cdots\}$. Is C closed under addition? Under subtraction? Explain your answers.

Notice that when we ask if a set is closed, we must also state what operation we are thinking of. The set of natural numbers is closed under addition, but it is not closed under subtraction. This property of the set of natural numbers and addition is expressed below.

The universe for a and b is N. For each replacement of a and b, there is a number $a+b$ that is a member of N.

? **L** $\{0, 2, 4, \cdots\}$ is closed under addition. Reword the property of the natural numbers that is expressed above so that what you say applies to $\{0, 2, 4, \cdots\}$ and addition.

On Your Own

For each of Exercises 1 through 4, tabulate the union of the two sets.

1 $A = \{1, 2, 3\}$.
$B = \{1, 3, 5\}$.

2 $A = \{0, 2, 4, 6, 8\}$.
$B = \{1, 2, 3, 4, 5\}$.

3 $A = \{0, 2, 4, 6, 8\}$.
$B = \{1, 3, 5, 7, 9\}$.

4 $A = \{6, 12, 18, 24, 30, \cdots\}$.
$B = \{8, 16, 24, 32, 40, \cdots\}$.

For each set named in Exercises 5 through 10, tabulate the union of that set and $\{2, 4, 6, 8, 10\}$.

5 $\{1, 2, 3\}$ **6** $\{2\}$

7 {1, 3} **8** {4, 8, 12, 16, ···}

9 {10, 15, 20} **10** {450}

11 For each set named in Exercises 5 through 10, tell whether that set and $\{2, 4, 6, 8, 10\}$ are disjoint sets.

12 Suppose that $5 = n(C)$ and $3 = n(D)$. What must you know before you can decide if you obtain a true statement from $8 = n(C \cup D)$?

Suppose that E, F, G, and H are disjoint sets, and that $8 = n(E)$, $20 = n(F)$, $19 = n(G)$, and $70 = n(H)$. Tell how many members are in each set named in Exercises 13 through 20.

13 $E \cup F$ **14** $F \cup E$

15 $G \cup E$ **16** $F \cup F$

17 $E \cup H$ **18** $(G \cup E) \cup F$

19 $E \cup (F \cup H)$ **20** $(E \cup F) \cup (G \cup H)$

21 If you use addition, onto what natural number should you map $(70,50)$; $(29,54)$; (x,x); $(x,2x)$; $(x,4x)$? Give examples to explain your last three answers. The universe for x is N.

22 If you use subtraction, onto what natural number should you map $(50,25)$; $(78,40)$; (x,x); $(2x,x)$; $(4x,x)$? Give examples to explain your last three answers. The universe for x is N.

23 Name the next five members of $\{3, 6, 9, 12, 15, \cdots\}$ that come after 15.

24 Is $\{3, 6, 9, 12, 15, \cdots\}$ closed under addition? Is this set closed under subtraction? Explain your answers.

For each set named in Exercises 25 through 30, tell whether or not the set is closed under addition. Tell whether or not the set is closed under subtraction.

25 the set of odd numbers

26 the set of even numbers less than 14

27 the set of all even numbers

28 the set of natural numbers greater than 50

29 the set of natural numbers less than 50

30 the set of odd numbers greater than 99

EXPLORING IDEAS
Lesson 5
Booklet 7

In this lesson you will learn about the product of two natural numbers and the quotient of two natural numbers.

You will remember from the last lesson that addition of natural numbers is a special many-to-one mapping in which each ordered pair of natural numbers is mapped onto the sum of its components. You will remember, too, that the set of natural numbers is closed under addition. In this lesson you will learn about multiplication of natural numbers.

We are giving careful attention to the meaning of addition and multiplication to help us prepare for an understanding of the natural number system.

A Three sets are tabulated in Display 1. Is set C the Cartesian set of A and B?

> *Reminder:* The Cartesian set $A \times B$ (read "A cross B") is the set of all ordered pairs whose first components are the members of A and whose second components are the members of B. (See page 7 of BOOKLET 3 or page 36 of BOOKLET 6.)

1

$A = \{0, 1\}$. $B = \{0, 1, 2\}$.

$C = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$.

B Read each sentence in Display 2. Does each sentence express a true statement?

2

$2 = n(A)$. $3 = n(B)$.

C The symbol at the right is read "the number of members in A cross B." Now read the sentence in Display 3 at the bottom of the page.

n(AXB)

D Does the sentence in Display 3 express a true statement?

E Look at Display 4. Tabulate $D \times E$.

4

$D = \{0, 1, 2, 3\}$.

$E = \{0, 1, 2, 3, 4, 5\}$.

F What natural number is associated with set D? With set E? With $D \times E$?

The natural number associated with a Cartesian set is the *product* of the numbers associated with the sets that are used to form the Cartesian set. The product is the number of members in the Cartesian set.

G The symbol 4×6 can be used to express the product of 4 and 6. Is $4 \times 6 = n(D \times E)$ a true statement? $D \times E$ is the set you tabulated in Exercise E.

H Look again at Display 1. Is 2×3 the product of the numbers associated with A and B? Is 2×3 the number associated with the Cartesian set of A and B?

I Suppose that a set A has 10 members and a set B has 5 members. Is the product 10×5 the number of members in $A \times B$?

J Suppose that A has 2 members and B has 7 members. Is the product 2×7 the number of members in $A \times B$?

Now let us use variables to help us make a definition of *product*. We will begin by letting S and T represent any two sets that are associated with natural numbers.

K If 5 is associated with S, and 6 is associated with T, then 5×6 is the number associated with what set?

L Now suppose that 9 is associated with S, and 3 is associated with T; then 9×3 is the number associated with what set?

?

M The universe for s and t is N. Let us agree that s is associated with set S and t is associated with set T. Then the product $s \times t$ is associated with what set?

product of two natural numbers (prod'əkt). The universe for s and t is N. Suppose that s is associated with set S, and t is associated with set T. The product $s \times t$ is the number associated with $S \times T$. Also, s and t are factors of the product $s \times t$.

3

$2 \times 3 = n(A \times B)$.

"Two times three" *"is equal to"* *"the number of members in"* *"A cross B."*

Reminder: The numbers we use to obtain a product are the factors of the product. (See page 88 of BOOKLET 6 or page 32 of BOOKLET 5.)

N Tabulate a set A so that $4 = n(A)$. Tabulate a set B so that $2 = n(B)$. Tabulate $A \times B$.

O The product 4×2 is associated with which of the sets you tabulated in Exercise N? What are two factors of the product 4×2?

P Suppose that A is the empty set and B is $\{2, 3, 4, 5\}$. What number is associated with A? With B?

Q What set is the Cartesian set of the empty set and $\{2, 3, 4, 5\}$? How do you know?

R Is 0 the product of 0 and 4? Use the empty set and $\{2, 3, 4, 5\}$ to explain your answer.

S You recall that in our work with sums we required that our two sets be disjoint. Is this requirement necessary in working with products? Explain your answer.

Now that you know what the product of two natural numbers is, you will learn something new about multiplication. You will learn that multiplication is a special kind of mapping.

A Look at Display 5. Have we mapped $(2,3)$ onto the product of 2 and 3? What did we do to map each ordered pair of natural numbers onto a natural number?

B We can think of the *operation of multiplication* of natural numbers as a special kind of mapping of ordered pairs of natural numbers onto natural numbers.

5

$(2,3)$	$(4,6)$	$(5,2)$	$(8,0)$	$(6,4)$
↓	↓	↓	↓	↓
2×3 or 6	4×6 or 24	5×2 or 10	8×0 or 0	6×4 or 24

Is each ordered pair mapped onto the natural number that is the product of the components of that pair?

C In multiplication, onto what number should we map $(8,2)$? $(4,4)$? $(1,16)$? How do you know that the multiplication mapping is not one-to-one? Name two other ordered pairs of natural numbers that are mapped onto 16 by multiplication.

D You know that both addition and subtraction are binary operations because we begin with two natural numbers. Using multiplication, we begin with $(5,3)$ and obtain what number? Using multiplication, we begin with $(4,20)$ and obtain what number?

E Is multiplication also a binary operation? Explain your answer.

Now you will learn about another binary operation that is related to multiplication in the same way that subtraction is related to addition.

F Look at Display 6. $(6,3)$ is mapped onto 2. Is the product of 2 and the second component of $(6,3)$ equal to the first component of $(6,3)$?

6

$(6,3)$	$(6,2)$	$(10,2)$	$(25,5)$
↓	↓	↓	↓
2	3	5	5
$2 \times 3 = 6$.	$3 \times 2 = 6$.	A	B

G What number is $(6,2)$ mapped onto? Multiply this number and the second

component of $(6,2)$. The product you obtain is what component of $(6,2)$?

H How have we used multiplication to map the ordered pairs of natural numbers named in Display 6 onto natural numbers? What sentence should be used to replace "box" A? "Box" B?

I You know that $3 \times 7 = 21$ is a true statement. How does this help you find the number that $(21,7)$ is mapped onto?

The number 3 that $(21,7)$ is mapped onto is the *quotient* of 21 divided by 7. The quotient of 21 divided by 7 is $21 \div 7$, or 3.

J Since the product of 5 and 3 is 15, the quotient of 15 divided by 3 is what number? The quotient of 15 divided by 5 is what number?

K Look again at Display 6. Is each ordered pair named there mapped onto the number that is the quotient of the first component of the pair divided by the second component? Explain your answer.

When an ordered pair is mapped onto the quotient of its first component divided by its second component, the mapping is the *operation of division.*

L In division, onto what number do we map $(8,2)$? $(14,7)$? $(18,18)$? $(5,1)$?

M What division mapping do we obtain from the statement $2 \times 5 = 10$? Explain your answer.

? **N** The universe for a, b, and c is N. Using the operation of division, we map (a,b) onto c only when the quotient of the replacement for a divided by the replacement for b is what number?

O Now you have seen how two natural numbers can be used to obtain a third number that is their quotient. If we begin with $(27,9)$, we obtain what quotient? If we begin with $(44,4)$, we obtain what quotient?

P Is division also a binary operation? How do you know?

? **Q** Is the operation of division a many-to-one mapping? Explain your answer.

We have thought of division as it is related to multiplication. $30 \div 10 = x$ means that we must find a replacement for x so that we obtain a true statement from $30 = x \times 10$. Now let us study more about multiplication and division of natural numbers.

A In multiplication, we map $(10,2)$ onto 20. In division, we map $(10,2)$ onto what natural number?

B In multiplication, we map $(3,4)$ onto what natural number?

C When we use division, we map $(3,4)$ onto a natural number so that the product of that number and 4 is the first component of $(3,4)$. Can we use division to map $(3,4)$ onto a natural number? Explain your answer.

D Can we use multiplication to map $(11,4)$ onto a natural number? Can we use division to map $(11,4)$ onto a natural number? Explain your answers.

E Is the product of two natural numbers always a natural number? Is the quotient of two natural numbers always a natural number?

You have learned about the set of natural numbers and the operations of multiplication and division. You learned that the product of any two members of N

is also a member of N. To express this in a shorter way, we say that N is *closed under multiplication.* N is closed under multiplication because the product of any two natural numbers is always a natural number.

You have also learned that the quotient of two natural numbers is *not* always a natural number. Therefore, N is *not* closed under division.

F The universe for a and b is N. When will the quotient of the replacement for a divided by the replacement for b be a natural number?

G Now describe each set tabulated in Display 7 below.

7

$A = \{0, 2, 4\}$.

$B = \{0, 2, 4, \cdots\}$.

H 0, 2, and 4 are members of A. Is 0×2 a member of A? Is 2×4 a member of A?

I Is set A closed under multiplication? How do you know?

J Is $4 \div 2$ a member of A ? Is $2 \div 4$ a member of A?

K Is set A closed under division? How do you know?

L Look at the tabulation of set B. Are 6, 12, and 20 members of B? Is 6×12 a member of B? Is 20×6 a member of B? Is B closed under multiplication?

M Is B closed under division?

N $C = \{1, 3, 5, 7, \cdots\}$. Is C closed under multiplication? Under division? Explain your answers.

You learned that the set of natural numbers is closed under multiplication. This property is expressed as follows:

The universe for a and b is N. For each replacement of a and b, there is a number $a \times b$ that is a member of N.

O $\{1, 3, 5, 7, \cdots\}$ is closed under multiplication. Reword the property of the natural numbers that is expressed above so that what you say applies to $\{1, 3, 5, 7, \cdots\}$ and multiplication.

On Your Own

In Exercises 1 through 9, $A = \{0, 1\}$; $B = \{5, 7, 9\}$; $C = \{10\}$; $D = \{\ \}$. Tabulate each of the Cartesian sets named.

1	$A \times B$	**4**	$B \times C$	**7**	$D \times A$
2	$B \times A$	**5**	$C \times D$	**8**	$B \times D$
3	$A \times A$	**6**	$A \times C$	**9**	$D \times D$

10 Using the sets A, B, C, and D named above, tell how many members are in A, in B, in C, and in D.

11 For each set that you tabulated in Exercises 1 through 9, tell how many members are in the set.

Suppose that $5 = n(E)$, $6 = n(F)$, $0 = n(G)$, and $100 = n(H)$. Tell how many members are in each of the sets named in Exercises 12 through 20.

12	$E \times F$	**15**	$H \times F$	**18**	$F \times G$
13	$F \times E$	**16**	$H \times H$	**19**	$H \times G$
14	$E \times E$	**17**	$E \times H$	**20**	$F \times F$

If you use multiplication, onto what natural number should you map each of the ordered pairs named in Exercises 21 through 26?

21	(70,35)	**23**	(20,3)	**25**	$(1,x)$
22	(50,3)	**24**	$(x,3)$	**26**	$(x,0)$

27 Give examples to explain your answers for Exercises 24, 25, and 26.

If you use division, onto what natural number should you map each of the ordered

pairs named in Exercises 28 through 33 below?

28 (50,25) **30** (10,1) **32** (x,x)

29 (35,7) **31** (4,4) **33** $(x,1)$

34 Give examples to explain your answers for Exercises 32 and 33.

For each set described in Exercises 35 through 39, tell whether or not the set is closed under multiplication. If not, explain your answer.

35 the set of odd numbers

36 the set of even numbers

37 the set of natural numbers greater than 20

38 the set of natural numbers less than 20

39 the set of natural numbers from 51 through 60

40 For each set described in Exercises 35 through 39, tell whether or not the set is closed under division. If not, explain your answer.

EXPLORING IDEAS
Lesson 6
Booklet 7

In this lesson you will learn that the order in which two natural numbers are added or multiplied does not affect the sum or product.

A Look at Display 1. The dots indicate that each list of sentences is endless. Does each sentence express a true statement?

1

$0 + 1 = 1 + 0.$
$0 + 2 = 2 + 0.$
$0 + 3 = 3 + 0.$
.....

$1 + 0 = 0 + 1.$
$1 + 2 = 2 + 1.$
$1 + 3 = 3 + 1.$
.....

$117 + 26 = 26 + 117.$
$117 + 27 = 27 + 117.$
.....

The statements expressed in Display 1 involve the natural numbers and the operation of addition. We know that these particular statements are true. We would like to know if every other possible statement of the kind expressed above is also true.

In this booklet we have used sets to help us think about numbers. Now we will use sets to help us decide if each possible statement is true.

B Look at Display 2. Describe each set tabulated. Are sets M and N disjoint? How do you know?

2

M = {Dick, Tony, Bob, Paul}.
N = {center, guard, quarterback}.

C What natural number is associated with set M? With set N?

D Tabulate $M \cup N$.

E Look at Display 3. Does the sentence express a true statement?

3

$4 + 3 = n(M \cup N).$

F Look again at Display 2. Tabulate $N \cup M$.

G Look at Display 4. Does the sentence express a true statement?

4

$3 + 4 = n(N \cup M).$

H Is the set that you tabulated in Exercise D the same as the set that you

tabulated in Exercise F? Is $M \cup N$ the same set as $N \cup M$?

> *Reminder*: Changing the order in which you list the names of members of a set does not change the set. (See pages 8 and 14 of BOOKLET 1 or page 4 of BOOKLET 6.)

I Does $M \cup N$ have the same number of members as $N \cup M$? Explain your answer.

J Is the sum *4+3* associated with the same set as the sum *3+4*?

K Are *4+3* and *3+4* the same number? Is $4 + 3 = 3 + 4$ a true statement? Explain your answers.

L Look at Display 5. Describe each set tabulated. Are sets D and E disjoint?

> **5**
>
> D = {1, 3, 5, 7, 9}.
>
> E = {10, 12, 14, 16, 18, 20}.

M What natural number is associated with set D? With set E?

N Tabulate $D \cup E$. Tabulate $E \cup D$. Are these sets the same?

O Look at Display 6. From your answers for Exercises M and N, how do you know that sentence A expresses a true statement? Does sentence B express a true statement?

> **6**
>
> **A** $5 + 6 = n(\text{D} \cup \text{E})$.
>
> **B** $6 + 5 = n(\text{E} \cup \text{D})$.

P Is the sum *5+6* associated with the same set as the sum *6+5*? Explain your answer. Is $5 + 6 = 6 + 5$ a true statement?

In Exercise H on the preceding page you noticed that $N \cup M$ is the same set as $M \cup N$. In Exercise N on this page you noticed that $E \cup D$ is the same set as $D \cup E$. From our definition of *union* on page 15 you know that the union of two sets is the set of all members of one set together with all members of the other.

Q Suppose that A and B are any two disjoint sets associated with natural numbers. Will $B \cup A$ be the same set as $A \cup B$? Explain your answer.

From Display 2 you observed that the number 4 is associated with set M, and the number 3 is associated with set N. Then you noted that the sum *4+3* is associated with the same set as the sum *3+4*.

From Display 5 you observed that the number 5 is associated with set D, and the number 6 is associated with set E. Then you noted that the sum *5+6* is associated with the same set as the sum *6+5*.

R Suppose that A and B are any two disjoint sets and that $a = n(A)$ and $b = n(B)$. The universe for a and b is the set of natural numbers. With what set is $a+b$ associated? With what set is $b+a$ associated? Do both $a+b$ and $b+a$ represent the number of members in the same set?

You can see that the order in which we add any two natural numbers does not affect the sum. We call this property the *commutative property of addition for natural numbers*. This property may be expressed as follows:

> **The universe for a and b is N. For each replacement of a and b, a true statement is obtained from $a + b = b + a$.**

S Look again at the bottom of page 29 and read the sentence expressing the commutative property. For each statement expressed in Display 7, name the replacements that have been made for the variables in $a + b = b + a$. Is each statement true? How do you know?

7

A $6 + 9 = 9 + 6$.

B $52 + 30 = 30 + 52$.

C $89 + 102 = 102 + 89$.

T Are $(6,9)$ and $(9,6)$ mapped onto the same natural number by addition?

U Are (a,b) and (b,a) mapped onto the same natural number by addition, for each replacement of a and b? How do you know?

V If a and b are each replaced by the same natural number, do we always obtain a true statement from $a + b = b + a$?

We have used the word *property* many times in this booklet. When we talk about property, we mean something that "belongs to." The commutative property, for example, "belongs to" addition for all natural numbers. In other words, the solution set of the condition $a + b = b + a$ is the set of all possible pairs of natural numbers.

A Now look at Display 8. Does each sentence express a true statement?

8

$0 \times 1 = 1 \times 0$.
$0 \times 2 = 2 \times 0$.
.....

$2 \times 0 = 0 \times 2$.
$2 \times 1 = 1 \times 2$.
.....

$56 \times 79 = 79 \times 56$.
$56 \times 80 = 80 \times 56$.
.....

The statements expressed in Display 8 involve the natural numbers and the operation of multiplication. We know that the statements are true. We will now use sets to help us decide if every other possible statement of the kind expressed in Display 8 is also true.

B In Display 9, describe each set tabulated.

9

$F = \{13, 14\}$.

$G = \{0, 1, 2\}$.

C What natural number is associated with set F? With set G?

D Tabulate $F \times G$. The first component of each member of $F \times G$ is a member of what set? The second component is a member of what set?

E Look at Display 10. Does the sentence express a true statement?

10

$2 \times 3 = n(F \times G)$.

F Look again at Display 9. Tabulate $G \times F$. The first component of each member of $G \times F$ is a member of what set? The second component is a member of what set?

G Look at Display 11. Does the sentence express a true statement?

11

$3 \times 2 = n(G \times F)$.

H Is the set you tabulated in Exercise D the same as the set you tabulated in Exercise F?

I Look at Display 12 at the top of the next page. Is $F \times G$ the same set that you tabulated in Exercise D? Is $G \times F$ the same set that you tabulated in Exercise F?

12

$F \times G = \{(13,0), (13,1), (13,2), (14,0), (14,1), (14,2)\}.$

$G \times F = \{(0,13), (1,13), (2,13), (0,14), (1,14), (2,14)\}.$

J For each member of $F \times G$, tell what member of $G \times F$ it is mapped onto. For each member of $G \times F$, tell what member of $F \times G$ it is mapped onto.

K What rule have we used in mapping the members of $F \times G$ onto the members of $G \times F$, and in mapping the members of $G \times F$ onto the members of $F \times G$?

L Are sets $F \times G$ and $G \times F$ in one-to-one correspondence? Is $F \times G$ equivalent to $G \times F$?

M Does $F \times G$ have the same number of members as $G \times F$? Explain your answer.

N Is $n(F \times G) = n(G \times F)$ a true statement? Is $2 \times 3 = 3 \times 2$ a true statement? Explain your answers.

A Look at Display 13. What natural number is associated with set J? With set K?

13

J = {3, 6, 9}.

K = {9, 12, 15, 18}.

B Tabulate $J \times K$. The first component of each member of $J \times K$ is a member of what set? The second component is a member of what set?

C Tabulate $K \times J$. The first component of each member of $K \times J$ is a member of what set? The second component is a member of what set?

D Are the sets that you tabulated in Exercises B and C the same set? Explain your answer.

E Make a diagram like the one in Display 12 to decide if $J \times K$ is equivalent to $K \times J$. Map each member of $J \times K$ onto the member of $K \times J$ that is obtained by interchanging the components of the member of $J \times K$.

F Look at Display 14. Does sentence A express a true statement? Does sentence B express a true statement?

14

A $3 \times 4 = n(J \times K)$.

B $4 \times 3 = n(K \times J)$.

G Is $n(J \times K) = n(K \times J)$ a true statement? How do you know? Is $3 \times 4 = 4 \times 3$ a true statement?

In Exercise L on this page you noticed that $F \times G$ is equivalent to $G \times F$. In Exercise E on this page you found that $J \times K$ is equivalent to $K \times J$. In both exercises, the sets were put in one-to-one correspondence by matching an ordered pair from one set with an ordered pair from the other set. One of the matched pairs could be obtained from the other by interchanging the components.

H Suppose that A and B are any two sets associated with natural numbers. Will $A \times B$ be equivalent to $B \times A$? Explain your answer.

From Display 9 you observed that the number 2 is associated with set F, and the number 3 is associated with set G. In Exercise N on this page you noted that the products 2×3 and 3×2 are equal since they are associated with equivalent sets.

From Display 13 you observed that the number 3 is associated with set J, and the number 4 is associated with set K. In Exercise G on page 31 you noted that the products 3×4 and 4×3 are equal since they are associated with equivalent sets.

I Suppose that A and B are any two sets and that $a = n(A)$ and $b = n(B)$. The universe for a and b is the set of natural numbers. With what set is $a\times b$ associated? With what set is $b\times a$ associated? Do $a\times b$ and $b\times a$ represent the number of members in equivalent sets? Do $a\times b$ and $b\times a$ represent the same number?

Now we know that the order in which we multiply any two natural numbers does not affect the product. We call this property the *commutative property of multiplication for natural numbers.* This property may be expressed as follows:

The universe for a and b is N.
For each replacement of a and b,
a true statement is obtained from
$a \times b = b \times a$.

Notice that we have used the symbol for multiplication ($\times$) in the open sentence $a \times b = b \times a$. In mathematics it is customary, in a sentence like this, to indicate the operation of multiplication by simply omitting the symbol for the operation. Thus, ab means the same as $a \times b$. We can, therefore, write the sentence $a \times b = b \times a$ as follows: $ab = ba$.

J You know that the symbol $2m$ tells us to find the product of 2 and the replacement for the variable m. Suppose that we replace m in $2m$ by 4. Then we obtain 2×4, or 8. You have already learned that you can write the numeral 2×4 as $2(4)$. Why do we need to use some symbol for multiplication in expressing a product such as $2(4)$?

When it is necessary to use a symbol to indicate multiplication, we often use a raised dot, as shown in Display 15.

K If sentence A in Display 15 did not contain symbols for multiplication, we might mistake the product $7\cdot8$ for 78. Are the numeral $7\cdot8$ and the numeral 78 names for the same number? How do you know?

15

A $7 \cdot 8 = 8 \cdot 7$.

B $29 \cdot 16 = 16 \cdot 29$.

C $115 \cdot 21 = 21 \cdot 115$.

L Look at the sentence in Column 1 expressing the commutative property for multiplication. For each statement expressed in Display 15, name the replacements that have been made for the variables in $ab = ba$. Is each statement true? How do you know?

M Are $(7,8)$ and $(8,7)$ mapped onto the same natural number by multiplication?

N Are (a,b) and (b,a) mapped onto the same natural number by multiplication, for each replacement of a and b? How do you know?

O If a and b are each replaced by the same natural number, do we always obtain a true statement from $ab = ba$?

Now let us see if subtraction of natural numbers also has the commutative property.

A Look at Display 16 at the top of the next page. What is the universe for the variables in the condition?

16

Universe for a and b is N.

$a - b = b - a$.

A $1 - 0 = 0 - 1$.

B $52 - 8 = 8 - 52$.

C $120 - 97 = 97 - 120$.

B Explain in words the condition expressed in Display 16. What replacements have been made for the variables in the condition to obtain each statement expressed?

Now we must decide if the statements expressed above are true.

C Look at sentence A. You know that *1–0* is the number 1. Do you think that *0–1* is also the number 1?

0–1 is not the number 1. Later on, you will learn that *0–1* is not a natural number. *8–52* and *97–120* are not natural numbers either.

D Is each statement expressed in Display 16 false? Explain your answer.

E Does subtraction of natural numbers have the commutative property? Explain your answer.

F How many statements like *52 – 8 = 8 – 52* do we have to inspect before we can decide that subtraction is not commutative? Explain your answer.

Now we will decide if division of natural numbers has the commutative property.

G Look at Display 17. What is the universe for the variables in the condition?

H Explain in words the condition expressed in Display 17. What replacements have been made for the variables in the condition to obtain each statement expressed?

17

Universe for a and b is N.

$a \div b = b \div a$.

A $12 \div 3 = 3 \div 12$.

B $96 \div 16 = 16 \div 96$.

C $144 \div 24 = 24 \div 144$.

I Is each statement expressed in Display 17 true? How do you know?

J Does division of natural numbers have the commutative property?

On Your Own

In Exercises 1 through 10, write "T" for each sentence that expresses a true statement. Write "F" for each sentence that expresses a false statement. Do not use computation to obtain your answers.

1 $119 + 46 = 46 + 119$.

2 $186 \div 3 = 3 \div 186$.

3 $42 \cdot 59 = 59 \cdot 42$.

4 $241 + 63 < 63 + 241$.

5 $23 - 18 = 18 - 23$.

6 $74 \cdot 16 > 16 \cdot 74$.

7 $(3 + 2) + 6 = (2 + 3) + 6$.

8 $8 + (9 + 12) = (12 + 9) + 8$.

9 $14_{five} + 32_{five} = 32_{five} + 14_{five}$.

10 $10_{two} \cdot 101_{two} = 101_{two} \cdot 10_{two}$.

For each condition expressed in Exercises 11 through 20, tabulate the solution set. The universe for the variables is N.

11 $36 + b = 15 + 36$.

12 $a \cdot 17 = 17 \cdot 5$.

13 $41 + b < 1 + 41$.

14 $16 + 13 = b + 16$.

15 $(9 + 4) + 7 = (b + 9) + 7$.

16 $2 \cdot 3 > 3 \cdot a$.

17 $12 + (8 + 5) = (5 + b) + 12$.

18 $54 \cdot b < 1 \cdot 54$.

19 $13 + b = b + 13$.

20 $a \cdot 24 = 24 \cdot a$.

Now we will invent a binary operation to mean the sum of the first number and twice the second number. We will use this symbol to indicate the new operation: * . We can express the meaning of this operation by writing the mathematical sentence below. The universe for a and b is N.

$a * b = a + 2b$.

21 To decide if operation "*" is commutative, we must determine the solution set of $a * b = b * a$. Let us replace a by 3 and b by 4. Now let us decide if $3 * 4 = 4 * 3$ is a true statement. Is $3 * 4 = 3 + 2(4)$ a true statement? Is $4 * 3 = 4 + 2(3)$ a true statement?

22 Is $3 + (2 \cdot 4) = 4 + (2 \cdot 3)$ a true statement?

23 Is $3 * 4 = 4 * 3$ a true statement?

24 Now replace a by 7 and b by 13. Decide if $7 * 13 = 13 * 7$ is a true statement.

25 Is operation "*" commutative?

Now we will decide if the set of natural numbers is closed under operation "*." We must decide if we obtain a natural number from $a*b$ for each replacement of a and b. The universe is N. What replacements for a and b have we used in the numbers expressed in Exercises 26 through 29?

26 $5 + 2(7)$

27 $9 + 2(4)$

28 $11 + 2(6)$

29 $19 + 2(35)$

30 Do you think the set of natural numbers is closed under operation "*"? Explain your answer.

For each of Exercises 31 and 32, the meaning of an operation is given. First, write a mathematical sentence that expresses the meaning. Next, decide if the operation is commutative. Then decide if the set of natural numbers is closed under the operation.

31 Operation "$\lightning$" means the sum of three times the first number and three times the second number.

32 Operation "$\bigtriangledown$" means four less than the sum of the first number and twice the second number.

EXPLORING IDEAS
Lesson 7
Booklet 7

In this lesson you will learn that numbers are grouped in pairs for the purposes of adding and multiplying.

A Look at Display 1. We must decide how to add when we are given three numbers in a definite order.

I
$4 + 3 + 6$

You have learned that addition is a binary operation and that addition maps a pair of numbers onto a number called the sum of the pair of numbers. To find the sum of more than two numbers, we must first work with a pair of numbers.

B Look at Display 2 at the top of the next page. In numeral A, the sum of what pair of numbers is expressed within the parentheses? Onto what natural number

2

A $(4 + 3) + 6$

B $7 + 6$

can the pair of numbers be mapped by addition?

C In numeral B, how was the number 7 obtained? Onto what natural number can $(7,6)$ be mapped by addition?

Now we have found the sum of 4, 3, and 6. Beginning with $(4,3)$, we obtained the sum $4+3$, or 7. Then we used $(7,6)$ and obtained the sum $7+6$, or 13.

D Now look at Display 3. We will decide if $4 + (3 + 6)$ is the same number as $(4 + 3) + 6$. In numeral A, the sum of what pair of numbers is expressed within the parentheses? Is this the same pair of numbers that you named in Exercise B?

3

A $4 + (3 + 6)$

B $4 + 9$

E Onto what natural number can $(3,6)$ be mapped by addition?

F Look at numeral B. How was the number 9 obtained? Onto what natural number can $(4,9)$ be mapped by addition?

G Is $(4 + 3) + 6 = 4 + (3 + 6)$ a true statement? How do you know?

Notice that we have used parentheses to help us think about grouping three numbers in pairs when we add. The symbol $(4 + 3) + 6$ indicates that we first added 4 and 3 and then added 6 to their sum. The symbol $4 + (3 + 6)$ indicates that we added 3 and 6 and then added their sum to 4.

H Look at Display 4. How do you know that each sentence expresses a true statement?

4

A $(14 + 8) + 5 = 14 + (8 + 5)$.

B $(52 + 20) + 19 = 52 + (20 + 19)$.

C $(112 + 13) + 1 = 112 + (13 + 1)$.

In Lesson 5 we used sets to help us decide that the order in which we add any two natural numbers does not change the sum that we obtain. Now we will use sets to help us decide if every other possible statement of the kind expressed in Display 4 is also true.

I Three sets are tabulated in Display 5. Is the number 4 associated with set F? What number is associated with set G? With set H?

5

$F = \{0, 1, 2, 3\}$.

$G = \{a, b, c, d, e\}$.

$H = \{+, -, \times, \div, <, >\}$.

J Are sets F, G, and H disjoint sets?

K Tabulate $(F \cup G) \cup H$. Which union should be tabulated first? Why? What should you do then?

L Tabulate $F \cup (G \cup H)$. Now which union should be tabulated first?

M Is the set that you tabulated in Exercise K the same as the set that you tabulated in Exercise L?

N What number is associated with $(F \cup G) \cup H$? With $F \cup (G \cup H)$?

O Is the number associated with $(F \cup G) \cup H$ the same as the number associated with $F \cup (G \cup H)$?

P Suppose that A, B, and C are any three disjoint sets associated with natural

numbers. Will $(A \cup B) \cup C$ be the same set as $A \cup (B \cup C)$? Explain your answer.

Q Suppose that A, B, and C are any three disjoint sets and that $a = n(A)$, $b = n(B)$, and $c = n(C)$. The universe for a, b, and c is N. With what set is $(a + b) + c$ associated? With what set is $a + (b + c)$ associated? Do both $(a + b) + c$ and $a + (b + c)$ represent the number of members in the same set?

You have seen in Displays 2, 3, and 4 that the way in which we grouped three numbers did not affect the sum. In Exercises I through Q we have asked questions suggesting how you might prove that the way in which you group any three natural numbers does not affect the sum.

As you continue in your study of mathematics, you will learn how to develop proofs. In Lesson 5 we began to indicate how this is done, and we continued to do this in the work above.

In mathematics we often accept certain statements as being true without proof. We use examples to help us decide which statements to accept. Then, as we acquire more knowledge of mathematics, we are able to prove some of these statements. Some statements are not proved, but are accepted as being true. We will accept the property that the way in which we group any three numbers does not affect the sum. We call this property the *associative property of addition for natural numbers*. This property may be expressed as follows:

The universe for a, b, and c is N. For each replacement of a, b, and c, a true statement is obtained from $(a + b) + c = a + (b + c)$.

R Look at the sentence in Column 1 expressing the associative property. For each statement expressed in Display 6, name the replacements that have been made for the variables in $(a + b) + c = a + (b + c)$. Is each statement true? How do you know?

6

A $(9 + 8) + 16 = 9 + (8 + 16)$.

B $(1 + 45) + 24 = 1 + (45 + 24)$.

C $(13 + 11) + 19 = 13 + (11 + 19)$.

S Look again at Display 1 on page 34. Explain why the symbol $4 + 3 + 6$ is accepted as a name of a number.

Now we will learn how to multiply when we are given three numbers.

A Look at Display 7. To find the product named by numeral A, we must first work with a pair of numbers.

7

A $2 \cdot 5 \cdot 8$

B $(2 \cdot 5)\, 8$

C $2\,(5 \cdot 8)$

B In numeral B, the product of what pair of numbers is expressed within the parentheses? Onto what natural number can $(2,5)$ be mapped by multiplication?

Notice that in numeral B we did not use a raised dot between the parentheses and the numeral 8. The parentheses tell us that 8 is to be multiplied by the product $2 \cdot 5$.

C Onto what natural number can $(10,8)$ be mapped by multiplication?

D In numeral C, the product of what pair of numbers is expressed within the

parentheses? Onto what natural number can $(5,8)$ be mapped by multiplication?

E Onto what natural number can $(2,40)$ be mapped by multiplication?

F Is the number you named in Exercise C the same as the number you named in Exercise E?

G Is $(2 \cdot 5)\ 8 = 2\ (5 \cdot 8)$ a true statement? How do you know?

Notice that we have used parentheses to help us think about grouping three numbers in pairs when we multiply. The symbol $(2 \cdot 5)\ 8$ indicates that we first multiplied 2 and 5 and then multiplied 8 by their product. The symbol $2\ (5 \cdot 8)$ indicates that we first multiplied 5 and 8 and then multiplied the product by 2.

H In Display 8, how do you know that each sentence expresses a true statement?

8

A $(7 \cdot 6)\ 14 = 7\ (6 \cdot 14)$.

B $(25 \cdot 18)\ 39 = 25\ (18 \cdot 39)$.

C $(123 \cdot 1)\ 14 = 123\ (1 \cdot 14)$.

You noticed in Display 8 that the way in which we grouped three numbers did not affect the product. At this time we will accept without proof that the way in which we group any three natural numbers does not affect the product. We call this property the *associative property of multiplication for natural numbers*. This property may be expressed as follows:

The universe for a, b, and c is N. For each replacement of a, b, and c, a true statement is obtained from $(ab)c = a(bc)$.

I Look at the sentence above expressing the associative property. For each statement expressed in Display 9, name the replacements that have been made for the variables in $(ab)c = a(bc)$. Is each statement true? How do you know?

9

A $(9 \cdot 12)\ 17 = 9\ (12 \cdot 17)$.

B $(82 \cdot 13)\ 5 = 82\ (13 \cdot 5)$.

C $(239 \cdot 4)\ 11 = 239\ (4 \cdot 11)$.

? **J** Look again at Display 7 on page 36. Explain why numeral A is accepted as a name of a number.

? **K** Show that the set $(A \times B) \times C$ is equivalent to the set $A \times (B \times C)$ for any three sets A, B, and C that are associated with natural numbers.

? **L** Use the results of Exercise K to explain why $(ab)c$ and $a(bc)$ represent the same number. The universe for a, b, and c is N.

Now let us see if subtraction of natural numbers has the associative property.

A Look at Display 10. What is the universe for the variables in the condition?

10

Universe for a, b, and c is N.

$(a - b) - c = a - (b - c)$.

$(4 - 3) - 1 = 4 - (3 - 1)$.

B Explain in words the condition expressed in Display 10. What replacements have been made for the variables to obtain the statement expressed?

C Is the statement true? The parentheses are used in the same way that they were used in addition and multiplication. They indicate which subtraction should be done first.

D Does subtraction of natural numbers have the associative property? Explain your answer.

Notice that on the basis of one example we have concluded that subtraction does not have the associative property. You may think that one example is not enough. But remember that if subtraction has the associative property, then the property must hold true for *all* natural numbers. We have shown one set of replacements for a, b, and c for which the property is *not* true. Hence, we know that it cannot be true for all natural numbers.

? **E** Can we conclude, on the basis of one example, that a property is true for all natural numbers?

? **F** Are the symbols $(4 - 3) - 1$ and $4 - (3 - 1)$ names of numbers? Can we accept the symbol $4 - 3 - 1$ as a name of a number? Explain your answers.

Now we will decide if division of natural numbers has the associative property.

G In Display 11, what is the universe for the variables in the condition?

11

Universe for a, b, and c is N.

$(a \div b) \div c = a \div (b \div c)$.

$(60 \div 6) \div 2 = 60 \div (6 \div 2)$.

H Explain in words the condition expressed in the display. What replacements have been made for the variables to obtain the statement expressed?

I Is the statement true? How do you know?

J Does division of natural numbers have the associative property?

K Explain why you can answer Exercise J by using just the example in Display 11.

? **L** Are the symbols $(60 \div 6) \div 2$ and $60 \div (6 \div 2)$ names of numbers? Can we accept the symbol $60 \div 6 \div 2$ as a name of a number? Explain your answers.

On Your Own

In Exercises 1 through 10, use the associative and commutative properties to decide whether each sentence expresses a true statement or a false statement. For each true statement, name the property upon which your answer depends.

1 $(43 + 16) + 9 = 43 + (16 + 9)$.

2 $34 \cdot 5 > 5 \cdot 34$.

3 $(17 - 4) - 2 = 17 - (4 - 2)$.

4 $(67 \cdot 15)\ 3 = 67\ (15 \cdot 3)$.

5 $44 \div 11 = 11 \div 44$.

6 $(14 \cdot 6)\ 8 < 14\ (6 \cdot 8)$.

7 $(13 + 5) + 7 = (5 + 13) + 7$.

8 $(10 \div 5) \div 2 = 10 \div (5 \div 2)$.

9 $143 + 20 = 20 + 143$.
five five five five

10 $102 - 21 = 21 - 102$.
three three three three

For each condition expressed in Exercises 11 through 20, tabulate the solution set. The universe for the variables is N.

11 $(81 + 14) + 10 = 81 + (14 + c)$.

12 $(a \cdot 12)\ 15 < 1\ (12 \cdot 15)$.

13 $27 + b = 19 + 27$.

14 $(a + 71) + 12 > 3 + (71 + 12)$.

15 $(44 \cdot b)\ 3 = (b \cdot 44)\ 3$.

16 $a \cdot 85 = 85 \cdot 12$.

17 $(19 + 21) + c = 11 + (19 + 21)$.

18 $(37 + b) + 14 = 37 + (b + 14)$.

19 $(1 + 0)\ 16 = (b + 1)\ 16$.

20 $(17 \cdot 23)\ 0 \neq 17\ (23 \cdot c)$.

21 Place parentheses in the numeral $48 \div 12 \div 2$ so that the numeral expresses the number 2.

22 Place parentheses in the numeral $21 - 14 - 7$ so that the numeral expresses the number 14.

23 Place parentheses in the numeral $54 \div 27 \div 9$ so that the numeral expresses the number 18.

24 Place parentheses in the numeral $18 - 9 - 3$ so that the numeral expresses the number 6.

For each of Exercises 25 through 30, explain, by using the associative and commutative properties, why the number named in Column B is the same as the number named in Column A. For example, you would answer Exercise 25 by writing the following sentences:

$37 + (76 + 92) = 37 + (92 + 76)$.

commutative property of addition

$37 + (92 + 76) = (37 + 92) + 76$.

associative property of addition

	A	B
25	$37 + (76 + 92)$	$(37 + 92) + 76$
26	$14\ (7 \cdot 9)$	$9\ (14 \cdot 7)$
27	$(41 + 7)\ 3$	$3\ (7 + 41)$
28	$101_{two} + (11_{two} + 10_{two})$	$(101_{two} + 10_{two}) + 11_{two}$
29	$(21 \cdot 19)\ 6$	$19\ (21 \cdot 6)$
30	$(24 + 18) + 12$	$(12 + 24) + 18$

EXPLORING IDEAS
Lesson 8
Booklet 7

In this lesson you will learn about a property that involves both multiplication and addition.

The associative and commutative properties involve just one operation. Now you will learn about a property that applies in some cases where multiplication and addition are used together. Remember that the properties that we are studying prepare us for an understanding of the natural number system.

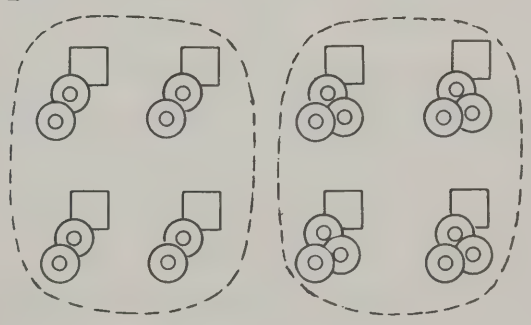
A Read the problem in Display 1.

1

Jean has exactly 8 record albums. She has 4 record albums containing 2 phonograph records each. She also has 4 albums containing 3 records each. How many records does Jean have?

B Look at Display 2. How are the record albums grouped? Does the picture show all Jean's records?

2



$4(2) + 4(3) = x$.

$U = N$.

C Does the open sentence in Display 2 express the condition for the problem in Display 1? For what is x a variable?

D Look at Display 3. Does Jean have *8+12* records? Tabulate the solution set of the condition.

3

$8 + 12 = x.$

Now we have found the solution set of the condition expressed in Display 2. We know that Jean has 20 phonograph records. First we used multiplication. Beginning with *(4,2)*, we obtained the product *4(2)*, or 8. Next, we used *(4,3)* and obtained the product *4(3)*, or 12. Finally, we used *(8,12)* and addition to obtain the sum *8+12*, or 20.

Now let us see if there is another way to obtain the solution set.

E Look again at the picture in Display 2. For each album containing 2 records, is there an album containing 3 records?

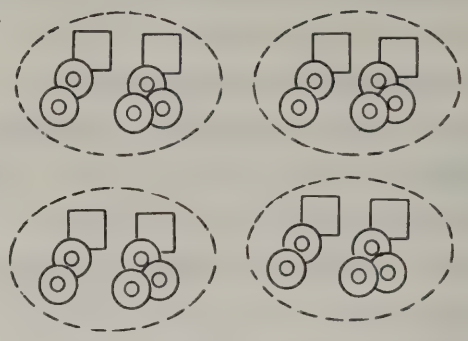
F We will regroup the albums pictured in Display 2. Suppose that each group of 3 records is joined with a group of 2 records. Will we now have *2+3* records in our new group? Altogether, how many groups containing *2+3* records will there be?

G Look at Display 4. Does each group contain *2+3* records? How many groups of albums are pictured? Has each record pictured in Display 2 been used in the regrouping in Display 4?

H Does the sentence in Display 4 express the condition for the problem in Display 1 after we have regrouped the albums? For what is x a variable?

I Look at Display 5. Does Jean now have 4 groups of albums, or 4 groups of 5 records? Tabulate the solution set of $4(5) = x$.

4



$4(2 + 3) = x.$

$U = N.$

5

$4(5) = x.$

J Is the set that you tabulated in Exercise I the same as the set that you tabulated in Exercise D?

K Is $4(2 + 3) = 4(2) + 4(3)$ a true statement? How do you know?

Notice that we have used parentheses to help us think about grouping numbers in pairs. The symbol *4(2+3)* indicates that we added 2 and 3 and then multiplied the sum by 4. The symbol $4(2) + 4(3)$ indicates that we first found the product of 4 and 2 and then the product of 4 and 3. Finally, we added the products.

Now let us consider another problem that involves both multiplication and addition.

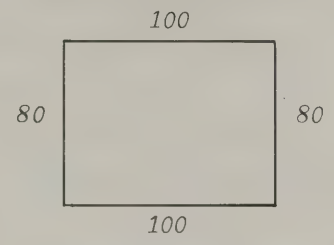
L Read the problem in Display 6.

6

A rectangular field is 100 yards long and 80 yards wide. The sum of the lengths of the sides of a field is the *perimeter*. What is the perimeter of this field?

M Display 7 shows a diagram of the field. How many 100-yard sides does the field have? How many 80-yard sides does it have?

7



$2(100) + 2(80) = x.$

N Does the sentence in Display 7 express the condition for the problem? For what is x a variable? The universe for x is N.

O Look at Display 8. Is the perimeter of the field *200+160* yards? Tabulate the solution set of the condition.

8

$200 + 160 = x.$

Now we know that the perimeter of the field is 360 yards. Let us use another way to find the perimeter.

P For each side of the field that is 100 yards long, is there a side 80 yards long?

Q Suppose that one side whose length is 80 yards is grouped with a side whose length is 100 yards. Will each new group of two sides contain *100+80* yards? How many groups of *100+80* will there be?

R Does sentence A in Display 9 express the condition for the problem in Display 6 after we have regrouped the lengths of the sides? For what is x a variable?

9

A $2(100 + 80) = x.$

B $2(180) = x.$

U = N.

S Look at sentence B. Does the perimeter of the field contain 2 groups of 180 yards? Tabulate the solution set of $2(180) = x.$

T Is the set that you tabulated in Exercise S the same as the set that you tabulated in Exercise O?

U Is $2(100 + 80) = 2(100) + 2(80)$ a true statement? How do you know?

Notice that we have used parentheses to help us think about grouping numbers in pairs. The symbol $2(100 + 80)$ indicates that we added 100 and 80 and then multiplied the sum by 2. The symbol $2(100) + 2(80)$ indicates that we first found the product of 2 and 100 and the product of 2 and 80. Then we added the products.

Both of the problems we studied were worked in two different ways. These ways help us understand how multiplication and addition are related. Now let us see if there is a property that relates multiplication and addition for all natural numbers.

A Look at Display 10. In sentence A, what is the common multiplier of the numbers 5 and 9? Does sentence A express a true statement?

10

A $6(5) + 6(9) = 6(5 + 9).$

B $10(3) + 10(7) = ___(3 + 7).$

C $9(18) + 9(4) = 9(___ + ___).$

B Look at sentence B. The number 10 is the common multiplier of what numbers? What numeral can you use to replace the blank so that sentence B expresses a true statement?

C Replace the blanks in sentence C in Display 10 so that it expresses a true statement.

D How do you know that each sentence in Display 11 expresses a true statement?

11

A $2(13 + 1) = 2(13) + 2(1)$.

B $8(10 + 7) = 8(10) + 8(7)$.

C $17(6 + 9) = 17(6) + 17(9)$.

You notice from sentence A in Display 11 that the result of multiplying the sum of 13 and 1 by 2 is the same as the result obtained by adding the product of 2 and 13 and the product of 2 and 1. In each sentence in Display 11, the product expressed at the left is the same number as the sum expressed at the right.

The statements expressed in Display 11 are examples of a property that relates multiplication and addition. The relationship is that we can multiply the sum of two numbers by a number, or we can multiply each of the two numbers used in getting the sum by the number and then add. In either case we get the same result.

After studying several examples, we will accept the property without proof. We call this property the *distributive property of multiplication over addition for natural numbers*. This simply means that multiplication is distributed over the two numbers used in obtaining a sum. The property may be expressed as follows:

The universe for a, b, and c is N. For each replacement of a, b, and c, a true statement is obtained from $a(b + c) = ab + ac$.

E The distributive property makes it possible for us to use two ways to find the number obtained from $a(b + c)$ after replacements. Describe each of these ways.

F Look at the sentence at the bottom of Column 1 expressing the distributive property. Then, for each statement expressed in Display 12, name the replacements that have been made for the variables in $a(b + c) = ab + ac$. Is each statement true? How do you know?

12

A $3(14 + 9) = 3(14) + 3(9)$.

B $16(12 + 8) = 16(12) + 16(8)$.

C $41(37 + 23) = 41(37) + 41(23)$.

D $3(3 + 3) = 3(3) + 3(3)$.

G Look at Display 13. You long ago learned to find the product of 6 and 53 by performing the multiplication expressed below.

13

$$\begin{array}{r} 53 \\ \times\ 6 \\ \hline 318 \end{array} \quad or \quad \begin{array}{r} 53 \\ \times\ 6 \\ \hline 18 \\ 300 \\ \hline 318 \end{array}$$

Now you will see that you are actually using the distributive property to obtain the product of 53 and 6.

H Look at Display 14. Does sentence A express a true statement? How do you know?

14

A $6 \times 53 = 6(50 + 3)$.

B $6(50 + 3) = 6(50) + 6(3)$.

C $6(50) + 6(3) = 300 + 18$.

D $300 + 18 = 318$.

I Does sentence B express a true statement? Upon what property does your answer depend?

J Do sentences C and D express true statements?

From the statements expressed in Display 14 you see how the distributive property is used to find the product of two natural numbers.

K Now use the distributive property to show that each of the following sentences expresses a true statement:

a $8 \times 47 = 376$. **c** $79 \times 3 = 237$.

b $7 \times 92 = 644$. **d** $35 \times 26 = 910$.

A Look at Display 15. Explain in words the condition expressed.

15

Universe for a, b, and c is N.

$(b + c)a = ba + ca$.

? **B** Read again the sentence on page 42 expressing the distributive property. How does $a(b + c) = ab + ac$ differ from $(b + c)a = ba + ca$?

C Look at Display 16. What replacements have been made for the variables in the condition $(b + c)a = ba + ca$ to obtain each statement expressed? Is each statement true? How do you know?

16

A $(6 + 8)13 = 6(13) + 8(13)$.

B $(15 + 19)4 = 15(4) + 19(4)$.

C $(7 + 5)83 = 7(83) + 5(83)$.

In Display 16 we have three examples in which true statements were obtained from the condition $(b + c)a = ba + ca$. Now let us see if this condition is satisfied by all possible replacements for the variables. We will use the properties that we already know to help us see this.

D Look at Display 17. For each replacement of a, b, and c, will the condition expressed by sentence A yield a true statement? We say that a condition "yields" a statement when the variables in the condition are replaced by members of the universe.

17

Universe for a, b, and c is N.

A $(b + c)a = a(b + c)$.

B $a(b + c) = ab + ac$.

C $ab + ac = ba + ca$.

E Upon what property does your answer for Exercise D depend?

F Does the condition expressed by sentence B yield a true statement for each replacement of a, b, and c? Upon what property does your answer depend?

G Does the condition expressed by sentence C yield a true statement for each replacement of a, b, and c? Upon what property does your answer depend?

H Does $(b + c)a = ba + ca$ yield a true statement for each replacement of a, b, and c? Explain your answer.

Now you have learned that multiplication distributes over addition and that the distributive property can be expressed in two forms. We can use the sentence $a(b + c) = ab + ac$ or the sentence $(b + c)a = ba + ca$, whichever we choose.

Next we will decide if addition distributes over multiplication.

? **A** Look at Display 18 on the next page. Explain in words the condition expressed.

B What replacements have been made for

the variables in the condition to obtain the statement expressed in Display 18?

18

Universe for a, b, and c is N.

$a + (bc) = (a + b)\ (a + c)$.

$3 + (4 \cdot 7) = (3 + 4)\ (3 + 7)$.

Now we will decide if the statement obtained from $a + (bc) = (a + b)\ (a + c)$ is true. If the statement is true, then the result of adding the product $4 \cdot 7$ to 3 is the same as adding each of the factors, 4 and 7, to 3 and then multiplying these sums.

C Does each sentence in Display 19 express a true statement?

19

A $3 + (4 \cdot 7) = 3 + 28$.

B $(3 + 4)\ (3 + 7) = 7 \cdot 10$.

D Is the statement expressed in Display 18 true? How do you know?

E Does addition distribute over multiplication for natural numbers?

? **F** How do we know that the condition expressed in Display 18 is not a property for natural numbers?

Now we will decide if multiplication distributes over subtraction for natural numbers.

G Look at Display 20. Explain in words the condition expressed here.

20

Universe for a, b, and c is N.

$a(b - c) = ab - ac$.

$5(6 - 2) = 5(6) - 5(2)$.

H What replacements have been made for the variables in the statement expressed?

I Is the statement expressed in Display 20 true? How do you know?

J Write two other sentences that express true statements obtained from $a(b - c) = ab - ac$. The universe for the variables is N.

? **K** From Display 20 you noticed that a true statement was obtained from $a(b - c) = ab - ac$. In Exercise J you obtained two other true statements from this condition. Do you think that we can accept as true the property that multiplication distributes over subtraction for all natural numbers? Explain your answer.

L Now look at Display 21. Explain in words each condition expressed.

21

Universe for a, b, and c is N.

A $a - (bc) = (a - b)\ (a - c)$.

B $a + (b - c) = (a + b) - (a + c)$.

C $a - (b + c) = (a - b) + (a - c)$.

D $a + (b \div c) = (a + b) \div (a + c)$.

M For each condition expressed in Display 21, write a sentence that expresses a statement obtained from the condition.

N Which sentences that you obtained in Exercise M express true statements? How do you know?

? **O** Does subtraction distribute over multiplication for natural numbers? Does addition distribute over subtraction? Does subtraction distribute over addition? Does addition distribute over division?

On Your Own

In Exercises 1 through 14, use the associative, commutative, and distributive properties to decide whether each sentence

expresses a true statement or a false statement. For each true statement, name the property that you used to make your decision.

1 $2(8) + 7(8) = (2 + 7)8$.

2 $19 + 73 = 73 + 19$.

3 $(16 \cdot 5)3 = 16(5 \cdot 3)$.

4 $7 + (8 \cdot 2) = (7 + 8)(7 + 2)$.

5 $4(21 + 9) = 4(21) + 4(9)$.

6 $(16 - 11)3 = 16(3) - 11(3)$.

7 $(13 + 8) + 4 = 13 + (8 + 4)$.

8 $13 \cdot 21 = 21 \cdot 13$.
four four four four

9 $14 - (1 + 3) = (14 - 1) + (14 - 3)$.

10 $(18 + 2) + 7 = (2 + 18) + 7$.

11 $12(6 - 2) = 12(6) - 12(2)$.

12 $15 - (2 \cdot 1) = (15 - 2)(15 - 1)$.

13 $6(3 \cdot 11) = 6(11 \cdot 3)$.

14 $14 + (4 \div 2) = (14 + 4) \div (14 + 2)$.

For each condition expressed in Exercises 15 through 26, tabulate the solution set. The universe for the variables is N.

15 $a(5 - 2) = 12$.

16 $n(14 - 6) < 32$.

17 $10 = x(3 + 2)$.

18 $a(5 - 2) = a(5) - a(2)$.

19 $y + 3 = 15(8 - 4)$.

20 $12x = 8x + 4x$.

21 $y(7 + 5) > 20$.

22 $30 - a = 14(3) - 14(2)$.

23 $k - 9 = 10(2 + 4)$.

24 $2h < 6(4) + 5(2)$.

25 $n(5 + 2) = n(5) + n(2)$.

26 $2x + 3x = 5x$.

For each of Exercises 27 through 36, a sum is expressed. Use the distributive property to express each of these sums as a product. For Exercise 27 you could write the following:

$15 + 35 = 5(3) + 5(7)$.

$5(3) + 5(7) = 5(3 + 7)$.

27 15 + 35

28 32 + 96

29 56 + 14

30 45 + 18

31 168 + 24

32 22 + 121

33 75 + 125

34 130 + 20

35 16 + 34

36 30 + 75

EXPLORING IDEAS
Lesson 9
Booklet 7

In this lesson you will learn about expressions and standard names for numbers.

A Numeral A in Display 1 tells us that we have obtained the number *9+7* by addition. What does each of the other numerals tell us?

1

A	9 + 7	**C**	19 - 3
B	8 · 2	**D**	32 ÷ 2

Each numeral in Display 1 contains a symbol for an operation. A numeral that contains one or more symbols for operations is an *expression.*

B You know that another name for the number *9+7* is the numeral 16. Does the numeral 16 contain a symbol for an operation?

The numeral 16 is the *standard name* of the number that can be expressed by such numerals as *9+7* and *32÷2.* A numeral is a standard name of a natural number if the numeral occurs in the list of symbols 0, 1, 2, 3, ···. Thus, the standard names that we commonly use are those that are

constructed according to the properties of our base-ten numeration system.

C Is the expression $32 \div 2$ the standard name of a number? Explain your answer.

Expressions are formed from standard names and symbols for operations.

D What standard names and operation symbol were used in forming the expression $8 \cdot 2$?

E For each expression in Display 1, give the standard name for the number.

In Exercise E you discovered that each expression in Display 1 is a name for the number 16. In your earlier work in arithmetic, you learned how to use computation to find standard names of numbers. For example, you used computation to find the standard name of $38 \cdot 75$.

Now let us determine the difference between an operation and computation. An operation assigns a number to an ordered pair of numbers. Thus, multiplication maps $(38, 75)$ onto $38 \cdot 75$. Now we have carried out the operation of multiplication. Computing enables us to find a standard name for $38 \cdot 75$. When we find the standard name 2850, we have used the multiplication process. The familiar addition, subtraction, multiplication, and division processes that you use in arithmetic enable you to find standard names of numbers.

In arithmetic most of your time has been spent in learning to compute. Thus, you worked with names of numbers. Now you are learning much more about the numbers themselves, their properties, and the properties of operations on numbers.

You have already learned how the symbol for "equals" is used with numerals. We write the sentence $38 \cdot 75 = 2850$ to express the statement that $38 \cdot 75$ and 2850 are the same number. In other words, the expression $38 \cdot 75$ and the standard name 2850 represent the same number.

F Look at expression A in Display 2. What operations have been performed? What numbers are involved in each operation?

2

A $4(5) + 4(3)$

B $6(7) - 6(2)$

G Give the standard name for the number represented by expression A.

H Is $4(5) + 4(3) = 32$ a true statement?

When we obtain the standard name of a number represented by an expression, we have *simplified* the expression.

I Look at expression B in Display 2. The expression includes what operation symbols?

J Simplify expression B.

K Now look at Display 3. What operation symbols does expression A include?

3

A $4 + 3 \cdot 6$

B $(4 + 3)6$

C $4 + (3 \cdot 6)$

Since expression A does not contain parentheses, we can think of two possible ways to simplify the expression. Expressions B and C indicate these two ways.

L The parentheses in expression B tell us that 6 has been multiplied by the sum $4+3$. Simplify expression B.

M The parentheses in expression C tell us that the product $3 \cdot 6$ has been added to 4. Simplify expression C.

N Is the standard name that you obtained in Exercise M the same as the standard name that you obtained in Exercise L?

It is clear that the product of $(4+3)$ and 6 is different from the sum of 4 and $(3 \cdot 6)$. For this reason, we must agree on the meaning of expressions that contain no parentheses. If we do not, we cannot accept the group of symbols $4 + 3 \cdot 6$ as a numeral, since it is not a name of just one number. In expressions that do not include parentheses, we will agree to give multiplication and division preference over addition and subtraction. We will agree that multiplication or division has been performed first. Of course, if the expression includes parentheses, the parentheses will indicate how to simplify the expression.

O Now we can simplify expression A. What is the standard name of the product $3 \cdot 6$?

P What is the standard name of the sum $4+18$?

Q Is $4 + 3 \cdot 6 = 22$ a true statement? How do you know?

R Look at expression A in Display 4. How do you know that 24 has been divided by 4 and that the quotient has then been added to 9? Simplify expression A.

4

A	$9 + 24 \div 4$	**D**	$3 \cdot 2 + 3 \cdot 3$
B	$4 \cdot 2 + 7$	**E**	$42 \div 2 - 18 \div 3$
C	$25 - 5 \cdot 2$	**F**	$3 \cdot 4 + 12 \div 4$

S Simplify each of the other expressions in Display 4.

T Are all the expressions in Display 4 names for the same number?

On Your Own

Exercises 1 through 4 give the standard names of four different numbers. For each number, write five names besides the standard name.

1 6 **3** 25

2 18 **4** 27

In each of Exercises 5 through 15, simplify the expression.

5 $9 + 6 \cdot 2$

6 $18 \div 6 + 7$

7 $8 \cdot 9 + 8 \cdot 3$

8 $16 - 5 \cdot 3$

9 $13 - 24 \div 8$

10 $(2 + 3)(2 + 7)$

11 $6(5 + 2) + 4 \cdot 3$

12 $7 + 8 \cdot 3 + 6$

13 $12 \cdot 5 - 45 \div 3$

14 $16 + (6 + 2)5$

15 $8(4 + 6)(3 + 7)$

In each of Exercises 16 through 27, tell whether the sentence expresses a true statement or a false statement.

16 $6(7) + 6(2) = 6(9)$.

17 $4(8 + 2) = 34$.

18 $18 + 7 \cdot 2 = (25)2$.

19 $4(17 - 3) = 56$.

20 $(16 + 5)6 = (21)6$.

21 $4 + 3(7 + 9) = (4 + 3)(7 + 9)$.

22 $48 \div 6 - 3 = 2 \cdot 8$.

23 $(5 + 8)(2 + 3) = 5(7 + 6)$.

24 $18 \div 3 + 21 \div 7 = 3(5) - 3(2)$.

25 $(14 + 9)4 + 2 = (23)6$.

26 $100 \div 10 - 5 = 5(4)$.

27 $72 \div 8 + 4 = 13$.

In each of Exercises 28 through 38, place parentheses in the sentence so that the sentence expresses a true statement.

28 $2 \cdot 5 + 6 = 16$.

29 $13 - 8 - 3 = 8$.

30 $5 + 4 \cdot 3 + 1 = 36$.

31 $60 = 2 \cdot 7 - 2 \cdot 6$.

32 $36 \div 9 + 3 \cdot 7 = 25$.

33 $3 \cdot 6 + 4 - 2 = 28$.

34 $9 = 14 \div 2 + 3 - 1$.

35 $6 + 3 \cdot 2 + 4 = 54$.

36 $8 = 18 + 2 \div 5 + 4$.

37 $42 \div 6 - 30 \div 5 = 1$.

38 $10 = 40 \div 5 + 3 - 1$.

EXPLORING IDEAS
Lesson 10
Booklet 7

In this lesson you will learn about the natural number system.

In this booklet you have learned how we obtain the set of natural numbers. You have studied the operations of addition and multiplication for natural numbers, and you are familiar with some of the properties of these operations. The aim of this lesson is to show how the *natural number system* is made up of the numbers, operations, and properties that you have been studying.

A Look at Display 1 at the top of the next page. Describe the set tabulated.

B Which properties expressed in Display 1 involve just addition? Which properties involve just multiplication? Remember that a property is a condition that yields a true statement for each permissible replacement of the variables.

C Which property expressed in Display 1 relates, or brings together, the operations of addition and multiplication?

D Each of the following sentences expresses a true statement. For each statement, name the property expressed in Display 1 upon which the truth of the statement depends.

a $(4 \cdot 3)5 = 4(3 \cdot 5)$.

b $6(2 + 7) = 6 \cdot 2 + 6 \cdot 7$.

c $15 + 39 = 39 + 15$.

d $(17 + 12) + 13 = 17 + (12 + 13)$.

e $6 \cdot 24 = 24 \cdot 6$.

f $35 \cdot 234$ is a natural number.

g $3594 + 72939$ is a natural number.

We are now able to give the definition of the system of natural numbers that we have been preparing you for.

> **system of natural numbers.** The system of natural numbers is made up of the set of natural numbers and the operations of addition and multiplication. The operations of addition and multiplication must be closed and be both commutative and associative. Multiplication must distribute over addition.

Notice from the definition that the system of natural numbers includes a set of objects, two operations, and certain properties of these operations. We say that any set of objects is a *number system* if it acts as the set of natural numbers does relative to two operations and the properties of these operations. The objects in such a set are called numbers.

As you continue your study of mathematics, you will learn about several

1

Universe = {0, 1, 2, ···}.

Operations:	Addition	Multiplication
Closure properties:	$a+b$ is a member of N.	ab is a member of N.
Commutative properties:	$a + b = b + a$.	$ab = ba$.
Associative properties:	$(a + b) + c = a + (b + c)$.	$(ab)c = a(bc)$.
Distributive property:	$a(b + c) = ab + ac$.	

number systems and study their properties. Now we will decide if another set forms a number system. To make this decision, we must decide if the properties that we have studied for addition and multiplication are satisfied by the members of the set.

A Look at Display 2. Is each member of set E a natural number? Is the number 2 a factor of each member of E? Describe the set tabulated.

2

E = {0, 2, 4, ···}.

Now we will see if E is closed under addition and multiplication. To do this, we must determine if $a+b$ and ab yield even numbers for each replacement of a and b from set E.

B Let us use 6 as a replacement for a, and 14 for b in both $a+b$ and ab. Are 6 and 14 even numbers? How do you know?

C Is $6+14$ an even number? Is 6×14 an even number?

D Select another replacement from E for a and b in $a+b$. Is the sum that you obtained an even number?

E Use the replacements from Exercise D for a and b in ab. Is the product that you obtained an even number?

F Do you think that E is closed under addition and multiplication? Explain your answer.

G Have we proved that $a+b$ and ab yield even numbers for each replacement of a and b from set E? Explain your answer.

H You know that 2 is a factor of every even number. Let $2m$ represent any even number and $2n$ another even number. The universe for m and n is N. Now use the distributive property to show that 2 is also a factor of the sum of any two even numbers.

I Show that the product of any two even numbers is also an even number.

J Now, using the results of Exercises H and I, can we say that we have proved that E is closed under addition and multiplication?

Next we will see if addition and multiplication are commutative in set E.

K Look at Display 3. For each of the statements expressed, what replacements have been made for the variables in the conditions? Is each statement true?

3

E = {0, 2, 4, ···}.

$a + b = b + a$.

A $2 + 8 = 8 + 2$.
B $22 + 34 = 34 + 22$.
C $74 + 18 = 18 + 74$.

$ab = ba$.

D $2 \cdot 8 = 8 \cdot 2$.
E $22 \cdot 34 = 34 \cdot 22$.
F $74 \cdot 18 = 18 \cdot 74$.

L Do you think that addition and multiplication are commutative in set E? Explain your answer.

Now we will decide if addition and multiplication are associative in the set of even numbers.

M Look at Display 4. For each of the statements expressed, what replacements have been made for the variables in the conditions? Is each statement true?

4

$E = \{0, 2, 4, \cdots\}$.

$(a + b) + c = a + (b + c)$.

A $(6 + 12) + 30 = 6 + (12 + 30)$.

B $(14 + 2) + 28 = 14 + (2 + 28)$.

$(ab)c = a(bc)$.

C $(18 \times 4) \times 10 = 18 \times (4 \times 10)$.

D $(32 \times 20) \times 8 = 32 \times (20 \times 8)$.

N Do you think that addition and multiplication are associative in set E? Explain your answer.

Now we will decide if multiplication distributes over addition in set E.

O Look at Display 5. For each of the statements expressed, what replacements have been made for the variables in the condition? Is each statement true?

5

$E = \{0, 2, 4, \cdots\}$.

$a(b + c) = ab + ac$.

A $4(12 + 8) = 4(12) + 4(8)$.

B $2(6 + 10) = 2(6) + 2(10)$.

C $56(24 + 18) = 56(24) + 56(18)$.

P Do you think that multiplication distributes over addition in the set of even numbers? Explain your answer.

We have concluded on the basis of several examples that, in the set of even numbers, addition and multiplication are closed; addition and multiplication are commutative and associative; and multiplication distributes over addition.

Q Is the set of even numbers a number system relative to addition and multiplication? Explain your answer.

R You know that both addition and multiplication are commutative and associative in the set of natural numbers. You also know that multiplication distributes over addition. Can you use these facts to determine if these properties hold true for subsets of the set of natural numbers? Explain your answer.

S Suppose you are working with subsets of the set of natural numbers. You wish to determine if these subsets form number systems with addition and multiplication. Is there a single property that you can examine for each operation to see if you have a number system? Explain your answer.

On Your Own

In Exercises 1 through 15, tell whether each sentence expresses a true statement or a false statement. Then, for each true statement, name the property of a natural number system that you used to make your decision.

1 $4 + 6 = 6 + 4$.

2 $(7 \cdot 9)11 = 7(9 \cdot 11)$.

3 $4(3 + 1) = 4(3) + 4(1)$.

4 $14 + 37$ is a natural number.

5 $7 + (6 \cdot 2) = (7 + 6)(7 + 2)$.

6 $16 \cdot 8 = 8 \cdot 16$.

7 $(5 + 12) + 1 = 5 + (12 + 1)$.

8 $0 + 13$ is a natural number.

9 $196 + 1 = 196$.

10 $17 + 35 = 35 + 17$.

11 $0 \cdot 16 = 16 \cdot 0$.

12 $(8 \cdot 11) + 4 = (8 \cdot 11) + (8 \cdot 4)$.

13 $7 \cdot 12$ is a natural number.

14 $(18 + 26) + 2 = 18 + (26 + 2)$.

15 $(14 \cdot 13)1 = 14(13 \cdot 1)$.

For each condition expressed in Exercises 16 through 24, tabulate the solution set. The universe for the variables is N.

16 $x(7 - 3) = 20$.

17 $7(4) + 8(4) = 65 - a$.

18 $y + 15 = 9(6 - 4)$.

19 $5 \cdot 7 - 4 > m(3 + 5)$.

20 $3k = 6 \cdot 2 + 4 \cdot 3$.

21 $2(5c) = (2 \cdot 5)c$.

22 $14x = 11x + 3x$.

23 $(19 + 14) + x = 19 + (x + 14)$.

24 $2 \cdot 8 - 3 \cdot 5 > x$.

For each set tabulated in Exercises 25 through 27, tell whether or not the set is a number system relative to addition and multiplication. If the set is *not* a number system, tell what property is not satisfied.

25 {1, 3, 5, ···}

26 {4, 8, 12, ···}

27 {3, 5, 6, 9, 10, 12, 15, 18, 20, ···}

EXPLORING IDEAS
Lesson 11
Booklet 7

In this lesson you will learn about other examples of number systems.

For several lessons now, we have been working with natural numbers and the system of natural numbers. The set of natural numbers is an infinite set.

Now we will decide if some sets that are not infinite also form number systems relative to two operations.

A Look at Display 1. How many members are in set M? Is set M an infinite set?

1

$M = \{E, O\}$.

We will use the letter *E* as a name for the set of even numbers, and the letter *O* as a name for the set of odd numbers. Thus, M is a set of two members. Each of the members, E and O, is a set in itself.

You know that in the system of natural numbers we need the operations of addition and multiplication. The operations of addition and multiplication that we will use for set M are not quite the same as those we use with the natural numbers.

B The two sentences in Display 2 show how we have agreed to think about addition and multiplication for set M. Sentence A expresses the statement that the sum of any even number and any odd number is an odd number. What statement is expressed by sentence B?

2

A $E + O = O$.

B $E \cdot E = E$.

C Look at Display 3 on the next page. The first table expresses the sum of any two members of M; the second table expresses the product of any two members.

D To find a name for the sum of E and O, you find the name in the space below the

3

+	E	O
E	E	O
O	O	E

×	E	O
E	E	E
O	E	O

letter *E* and to the right of the letter *O*. What member of M is named here?

E The sum of what other members of M is also O?

F To find a name for the product of O and E, you find the name in the space below the letter *O* and to the right of the letter *E*. What member of M is named here?

G The product of what other members of M is also E?

Now we have a set M and two operations on M. Next we will decide if both operations are closed; if both operations are commutative and associative; and if one operation distributes over the other.

H Is the sum of any two members of M also a member of M? Is the product of any two members of M also a member of M?

I Is set M closed under the operation of addition? Is set M closed under the operation of multiplication?

We must also decide if addition and multiplication are commutative in set M.

J Look at Display 4. We will use the addition table in Display 3 to decide if sentence A expresses a true statement.

K What member of M is the sum of E and O? What member of M is the sum of O and E?

L What replacements were made from set M for the variables in $a + b = b + a$ to obtain the statement expressed by sentence A? Does the sentence express a true statement? How do you know?

4

$M = \{E, O\}$.

$a + b = b + a$.

A $E + O = O + E$.

B $E + E = E + E$.

C $O + O = O + O$.

D $O + E = E + O$.

$ab = ba$.

E $E \times O = O \times E$.

F $E \times E = E \times E$.

G $O \times O = O \times O$.

H $O \times E = E \times O$.

M Do sentences B, C, and D express true statements? How do you know?

N How can you use the multiplication table in Display 3 to decide if sentence E expresses a true statement? What replacements from set M were made for the variables in $ab = ba$ to obtain the statement?

O Do sentences F, G, and H express true statements? How do you know?

? **P** Do the sentences in Display 4 express all possible statements obtained from $a + b = b + a$ when the members of set M are used as replacements for the variables?

Q Is the operation of addition commutative in set M? Is the operation of multiplication commutative in set M? Explain your answers.

Now we will decide if addition and multiplication are associative in set M.

A Look at sentence A in Display 5 on the next page. We will use the addition table in Display 3 to help us simplify each expression in the sentence. To simplify $(E+O)+E$, we must first find the sum of E and O. Is O the sum of E and O? How do you know? Next, we will add E to the sum of E and O. What member of M do we obtain?

B Is $(E + O) + E = O$ a true statement? How do you know?

5

$M = \{E, O\}$.

$(a + b) + c = a + (b + c)$.

A $(E + O) + E = E + (O + E)$.

B $(O + E) + O = O + (E + O)$.

C $(E + O) + O = E + (O + O)$.

D $(O + E) + E = O + (E + E)$.

E $(E + E) + O = E + (E + O)$.

F $(O + O) + E = O + (O + E)$.

G $(E + E) + E = E + (E + E)$.

H $(O + O) + O = O + (O + O)$.

C Next we will simplify the expression $E+(O+E)$ in sentence A. First we must find the sum of O and E. Is O the sum of O and E? How do you know?

D Now we will add to E the sum that we obtained in Exercise C. What member of M do we obtain?

E Is $E + (O + E) = O$ a true statement? How do you know?

F Is $(E + O) + E = E + (O + E)$ a true statement? How do you know?

G What replacements from set M were made for the variables in $(a + b) + c = a + (b + c)$ to obtain the statement?

H How do you know that each of the other sentences in Display 5 expresses a true statement?

I Do the sentences in Display 5 express all possible statements obtained from $(a + b) + c = a + (b + c)$ when the members of set M are used as replacements for the variables?

J Is the operation of addition associative in set M? How do you know?

K Now look at Display 6. Use the multiplication table in Display 3 to simplify each expression in the sentence $(E \times O)E = E(O \times E)$.

6

$M = \{E, O\}$.

$(ab)c = a(bc)$.

$(E \times O)E = E(O \times E)$.

L Is $(E \times O)E = E$ a true statement? Is $E(O \times E) = E$ a true statement? Explain your answers.

M Is $(E \times O)E = E(O \times E)$ a true statement? How do you know?

N What replacements were made for the variables in $(ab)c = a(bc)$ to obtain the statement?

O To be certain that multiplication is associative in set M, we must decide if all possible statements obtained from $(ab)c = a(bc)$ are true. Write the seven other sentences that express statements obtained from $(ab)c = a(bc)$. Use the members of M as replacements for the variables.

P Is each statement that you obtained in Exercise O true? How do you know?

Q Is the operation of multiplication associative in set M?

Finally, we will decide if multiplication distributes over addition in set M.

A Look at sentence A in Display 7. Use the tables in Display 3 to simplify each expression in the sentence.

7

$M = \{E, O\}$.

$a(b + c) = ab + ac$.

A $E(E + O) = (E \times E) + (E \times O)$.

B $O(E + O) = (O \times E) + (O \times O)$.

C $E(O + O) = (E \times O) + (E \times O)$.

D $O(E + E) = (O \times E) + (O \times E)$.

B Is $E(E + O) = E$ a true statement? Is $(E \times E) + (E \times O) = E$ a true statement? Explain your answers.

C Is $E(E + O) = (E \times E) + (E \times O)$ a true statement? How do you know?

D What replacements were made for the variables in $a(b + c) = ab + ac$ to obtain the statement?

E How do you know that each of the other three sentences in Display 7 expresses a true statement?

F Write the four other possible sentences that express statements obtained from $a(b + c) = ab + ac$.

G Is each statement that you obtained in Exercise F a true statement? How do you know?

H Does multiplication distribute over addition in set M?

We have seen that set M is closed under the operations of addition and multiplication; that addition and multiplication are commutative and associative; and that multiplication distributes over addition.

I Is set M a number system relative to addition and multiplication? Explain your answer.

J Explain why we can now refer to E and O as numbers.

K Now look at sentence A in Display 8. The sentence tells us that the result of dividing 13 by 5 is the quotient 2 and a remainder of 3. Does sentence A express a true statement?

L Does each of the other sentences in Display 8 express a true statement?

M Tabulate the set whose members are the remainders expressed in Display 8.

8

A $13 \div 5 = 2$ and a remainder of 3.

B $146 \div 5 = 29$ and a remainder of 1.

C $89 \div 5 = 17$ and a remainder of 4.

D $112 \div 5 = 22$ and a remainder of 2.

E $65 \div 5 = 13$ and a remainder of 0.

N Look at Display 9. Is the set tabulated here the same set that you tabulated for Exercise M?

9

$R = \{0, 1, 2, 3, 4\}$.

O Select any three natural numbers. Divide each of these numbers by 5. For each number tell what quotient and what remainder you obtained.

P Is each of the remainders that you obtained in Exercise O a member of set R?

Q Does set R contain all possible remainders that can be obtained by dividing any natural number by 5? Explain your answer.

R Is set R an infinite set?

We will decide if set R is a number system relative to two operations. The operations of addition and multiplication that we will use on set R, however, are different from the ordinary operations of addition and multiplication.

A Look at Display 10. What statement is expressed by sentence A?

10

$R = \{0, 1, 2, 3, 4\}$.

A $4 + 3 = 2$.

B $3 + 2 = 0$.

C $2 + 4 = 1$.

D $4 + 4 = 3$.

You know that ordinary addition maps $(4,3)$ onto the number that is the sum of the components. In our new addition, we first map $(4,3)$ onto the number that is the sum of the components. We then divide this sum by 5 and use the remainder for our new sum of 4 and 3.

B Is the number 7 the ordinary sum of 4 and 3? When we divide 7 by 5, do we obtain the quotient 1 and a remainder of 2?

C How can we use our new operation of addition to decide if sentence A expresses a true statement?

D How do you know that each of the other sentences in Display 10 expresses a true statement?

E Display 11 contains a table that expresses the sum of any two members of R. To find a name for the sum of 4 and 1, you find the name in the space below the numeral 4 and to the right of the numeral 1. What member of R is named here?

11

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

F Are there other pairs of members whose sum is also 0? Name them. How did you obtain your answers?

G Is the sum of any two members of R also a member of R? Is set R closed under our new operation of addition? How does the table in Display 11 help you decide that set R is closed under addition?

Now we will see how multiplication in set R differs from ordinary multiplication.

H Look at Display 12. What statement is expressed by sentence A?

12

R = {0, 1, 2, 3, 4}.

A $2 \times 3 = 1$.

B $4 \times 2 = 3$.

C $3 \times 4 = 2$.

You know that ordinary multiplication maps $(2,3)$ onto the number that is the product of the components. In our new multiplication, we first map $(2,3)$ onto the ordinary product. We then divide this product by 5 and use the remainder for our new product of 2 and 3.

I When we divide 6 by 5, do we obtain the quotient 1 and a remainder of 1? How can we use our new operation of multiplication to decide if sentence A expresses a true statement?

J How do you know that each of the other sentences in Display 12 expresses a true statement?

K Look at Display 13. To find a name for the product of 1 and 3, you find the name in the space below the numeral 1 and to the right of the numeral 3. What member of R is named in this space? Why is 3 the product of 1 and 3?

13

×	0	1	2	3	4
0	0				
1	0	1			
2	0	2	4		
3	0	3	1	4	
4	0	4	3	2	1

L Copy and complete the table shown in Display 13.

M Are there other pairs of members of R whose product is the same as the number that you obtained in Exercise K? Name them.

N Is the product of any two members of R also a member of R? Is set R closed under the operation of multiplication?

O How does the table that you made for Exercise L help you decide that set R is closed under multiplication?

Now we have a set R and two operations that are closed under R. We will decide if addition is commutative in set R.

P Look at sentence A in Display 14. We will use the table in Display 11 to help us decide if the sentence expresses a true statement.

14

$R = \{0, 1, 2, 3, 4\}.$

$a + b = b + a.$

A $4 + 1 = 1 + 4.$

B $2 + 4 = 4 + 2.$

C $3 + 4 = 4 + 3.$

Q What member of R is the sum of 4 and 1? What member of R is the sum of 1 and 4?

R Does sentence A express a true statement? How do you know?

S Does each of the other sentences in Display 14 express a true statement? How do you know?

T Do the sentences in Display 14 express all possible statements obtained from $a + b = b + a$? Remember, the universe for the variables is set R.

U If you use members of set R as replacements for the variables in $a + b = b + a$, is it possible to write a sentence that expresses a false statement?

V Do you think that addition is commutative in set R?

W The table in Display 15 below is the same as the table shown in Display 11. The numerals in dark type make up the *diagonal* of the table. This diagonal extends from the upper left to the lower right. Find the sum of 1 and 2. Is this sum expressed below the diagonal?

15

+	0	1	2	3	4
0	**0**	1	2	3	4
1	1	**2**	3	4	0
2	2	3	**4**	0	1
3	3	4	0	**1**	2
4	4	0	1	2	**3**

X Find the sum of 2 and 1. Where is the name of this sum located in the table in Display 15?

Y How do the positions of the numerals in the table show that addition is commutative in set R?

Now we will decide if multiplication is commutative in set R.

A Look at sentence A in Display 16. We will decide if the sentence expresses a true statement. Use the table you completed for Exercise L on this page.

16

$R = \{0, 1, 2, 3, 4\}.$

$ab = ba.$

A $3 \times 2 = 2 \times 3.$

B $4 \times 1 = 1 \times 4.$

C $3 \times 4 = 4 \times 3.$

B What member of R is the product of 3 and 2? What member of R is the product of 2 and 3?

C Does sentence A express a true statement? How do you know?

D Does each of the other sentences in Display 16 express a true statement? How do you know?

E When you replace the variables in $ab = ba$ by members of R, is it possible to write a sentence that expresses a false statement?

F Do you think that multiplication is commutative in set R? Explain your answer.

? **G** How do the positions of the numerals in the table that you made for Exercise L on page 56 show that multiplication is commutative in set R?

We will decide if addition and multiplication are associative in set R.

H Look at sentence A in Display 17. We will use the table in Display 15, page 56, to help us decide if the sentence expresses a true statement.

17

$R = \{0, 1, 2, 3, 4\}$.

$(a + b) + c = a + (b + c)$.

A $(4 + 2) + 3 = 4 + (2 + 3)$.

B $(1 + 3) + 4 = 1 + (3 + 4)$.

$(ab)c = a(bc)$.

C $(4 \cdot 2)3 = 4(2 \cdot 3)$.

D $(1 \cdot 3)4 = 1(3 \cdot 4)$.

I To simplify the expression $(4+2)+3$, we will first find the sum of 4 and 2. Is 1 the sum of 4 and 2? How do you know? Now we will add 3 to this sum. What number do we obtain?

J Is $(4 + 2) + 3 = 4$ a true statement? How do you know?

K Next we will simplify the expression $4+(2+3)$. First, we will find the sum of 2 and 3. What number do we obtain? Next we will add this sum to 4. What number do we obtain?

L Does $4 + (2 + 3) = 4$ express a true statement? How do you know?

M Does sentence A express a true statement? How do you know?

N Does sentence B express a true statement? How do you know?

? **O** How many possible statements can you obtain from $(a + b) + c = a + (b + c)$ by using the members of R as replacements for the variables?

P Do you think that each statement obtained from $(a + b) + c = a + (b + c)$ is true?

Q When you use members of R as replacements for the variables in $(a + b) + c = a + (b + c)$, is it possible to write a sentence that expresses a false statement?

R Do you think that addition is associative in set R? Explain your answer.

S Now use the multiplication table that you made for Exercise L, page 56, to decide if sentences C and D express true statements. Does each of these sentences express a true statement? How do you know?

T When you use members of R as replacements for the variables in $(ab)c = a(bc)$, is it possible to write a sentence that expresses a false statement?

U Do you think that multiplication is associative in set R? Explain your answer.

Now we will decide if multiplication distributes over addition in set R.

V Does each sentence in Display 18 express a true statement?

18

$R = \{0, 1, 2, 3, 4\}$.

$a(b + c) = ab + ac$.

A $2(4 + 3) = 2(4) + 2(3)$.

B $3(2 + 1) = 3(2) + 3(1)$.

C $2(0 + 4) = 2(0) + 2(4)$.

W When you use members of R as replacements for the variables in $a(b + c) = ab + ac$, is it possible to write a sentence that expresses a false statement?

X Do you think that multiplication distributes over addition in set R? Explain your answer.

We have studied many examples that show us that set R is closed under the operations of addition and multiplication; that both operations are commutative and associative; and that multiplication distributes over addition. Since we were unable to obtain any false statements from the properties, we will accept these properties for all members of R.

Y Is set R a number system relative to our new operations of addition and multiplication? Explain your answer.

Set R forms a very interesting number system. While there are only five members in the entire system, this system has important applications in modern physics and chemistry.

You remember that the set of natural numbers is not closed under subtraction. Let us decide if set R is closed under subtraction.

A Look at Display 19. To find the number onto which $(2,4)$ can be mapped by subtraction, we must find the replacement for x that satisfies the condition expressed by sentence A. *"$2 - 4 = x$"* means you are to find the replacement for x that enables you to obtain a true statement from the condition expressed by sentence B.

19

$R = \{0, 1, 2, 3, 4\}$.

$(2,4)$

A $2 - 4 = x$. **B** $2 = x + 4$.

B Look again at Display 15 on page 56. Is the sum of 3 and 4 the number 2? Onto what member of R can $(2,4)$ be mapped by subtraction?

C Onto what member of R can each of the following ordered pairs be mapped by subtraction?

$(2,3)$ $(1,4)$ $(4,3)$

D Can you think of an ordered pair whose components are members of R and whose difference is not a member of R?

E Do you think that set R is closed under subtraction? Explain your answer.

F Look at Display 20. To find the number onto which $(2,4)$ can be mapped by division, we must find the replacement for x that satisfies the condition expressed by sentence A. *"$2 \div 4 = x$"* means you are to find the replacement for x that enables you to obtain a true statement from $2 = x \cdot 4$.

20

$R = \{0, 1, 2, 3, 4\}$.

$(2,4)$

A $2 \div 4 = x$. **B** $2 = x \cdot 4$.

G Now use the multiplication table that you completed for set R. Is the product of 3 and 4 the number 2? Onto what member of R can $(2,4)$ be mapped by division?

H Onto what member of R can each of the following ordered pairs be mapped by division?

$(3,1)$ $(4,3)$ $(2,3)$

I Do you think that set R is closed under division? Explain your answer.

J Look at Display 21. How many members are in set S? The results of performing the operations * and # are shown in the tables. Use the tables to decide if set S is a number system relative to * and #.

21

$S = \{c, d\}$.

*	c	d
c	c	c
d	d	c

#	c	d
c	c	d
d	d	c

In this lesson and the preceding one, you have been working with different examples of number systems. For each example you noticed that you had a set, two operations, and some common properties. We say that these different examples have a common *structure*. You will learn more about the structure of mathematical systems as you continue your study of mathematics.

On Your Own

1 Set F is tabulated in Display 22 in the next column. Does set F contain all possible remainders that can be obtained by dividing any natural number by 3?

2 Is set F closed under addition? Under multiplication? Explain your answers.

22

$F = \{0, 1, 2\}$.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

3 Are addition and multiplication commutative in set F? Explain your answer.

4 Are addition and multiplication associative in set F? Explain your answer.

5 Does multiplication distribute over addition in set F? Explain your answer.

6 Is set F a number system relative to the operations of addition and multiplication? Explain your answer.

7 Set G is tabulated in Display 23. Does set G contain all possible remainders that can be obtained by dividing any natural number by 7?

23

$G = \{0, 1, 2, 3, 4, 5, 6\}$.

+	0	1	2	3	4	5	6
0							
1							
2			4				
3					0		
4							
5							
6				2			

8 To add members of G, first find the ordinary sum of two members of G. Next, divide the sum by 7 and use the remainder for the new sum. Copy and complete the table in Display 23.

9 Is set G closed under addition? Explain your answer.

10 Is addition commutative and associative in set G? Explain your answer.

11 Look at Display 24. To multiply members of G, first find the ordinary product of two members of G. Next, divide the product by 7 and use the remainder for the new product. Copy and complete the table in the display.

24

$G = \{0, 1, 2, 3, 4, 5, 6\}$.

×	0	1	2	3	4	5	6
0							
1							
2							5
3			6				
4							
5						4	
6							

12 Is set G closed under multiplication? Explain your answer.

13 Is multiplication commutative and associative in set G? Explain your answer.

14 Does multiplication distribute over addition in G? Explain your answer.

15 Is set G a number system relative to the operations of addition and multiplication? Explain your answer.

For each of the conditions expressed in Exercises 16 through 23, tabulate the solution set. The universe for the variables is set F, tabulated in Display 22 on the preceding page.

16 $1 - 2 = x$.

17 $1 - 0 = x$.

18 $2 - 0 = x$.

19 $2 - 1 = x$.

20 $0 - 2 = x$.

21 $1 \div 2 = x$.

22 $2 \div 1 = x$.

23 $0 \div 1 = x$.

EXPLORING IDEAS
Lesson 12
Booklet 7

In this lesson you will learn that the natural numbers zero and one possess special properties.

You have learned about the properties required for a number system. In this lesson you will learn that a number system may have additional properties.

A Look at Display 1. What natural number is associated with set E? With set F?

1

$E = \{\ \}$.

$F = \{5, 6, 7, 8\}$.

B Tabulate $E \cup F$. Is the sum $0 + 4$ associated with $E \cup F$? How do you know?

C Is set F the same as the set that you tabulated in Exercise B?

D Is $0 + 4 = 4$ a true statement?

E Look at the sets tabulated in Display 2. What natural number is associated with set C? With set D? With set E?

2

$C = \{1, 3, 5, 7, 9\}$.

$D = \{10, 11, 12\}$.

$E = \{\ \}$.

F Tabulate $E \cup C$. Is the sum $0 + 5$ associated with this set? How do you know?

G Tabulate $E \cup D$. What sum is associated with this set? How do you know?

H Is set C the same as set $E \cup C$, which you tabulated in Exercise F? Is $0 + 5 = 5$ a true statement?

I Is set D the same as set $E \cup D$, which you tabulated in Exercise G? Is $0 + 3 = 3$ a true statement?

In Exercise C you noticed that $E \cup F$ is the same set as F. In Exercise H you noticed that $E \cup C$ is the same set as C. From our definition of *union* you know that the union of two sets is the set of all the members of one set together with all the members of another set.

J Suppose that A is any set associated with a natural number. Will $\{\ \} \cup A$ be the same as set A? Explain your answer.

K You know that $0 + 4 = 4$, $0 + 5 = 5$, and $0 + 3 = 3$ are true statements. Now suppose that A is any set associated with a natural number, and that $a = n(A)$. Remember that we read $a = n(A)$ as "a equals the number of members in set A." You know that $0 = n\{\ \}$. With what set is $0 + a$ associated? Do both $0 + a$ and a represent the number of members in the same set?

Now we know that the sum of 0 and any natural number is identical with the natural number. For this reason, we call the number 0 the *identity element for addition.*

Since the sum of 0 and any natural number is the natural number itself, we call this property the *identity-element property for addition of natural numbers.* This property may be expressed as follows:

The universe for a is N. For each replacement of a, a true statement is obtained from $0 + a = a$.

L Do you think that some other member of N besides zero is also an identity element for addition? Explain your answer.

M Look at Display 3. For each statement expressed, tell what replacement has been made for the variable a in $0 + a = a$. How do you know that each sentence expresses a true statement?

3

U = N.

$0 + a = a$.

A $0 + 8 = 8$.

B $0 + 99 = 99$.

C $0 + 0 = 0$.

N Is the condition that is expressed in Display 4 a property? Remember that if a condition is a property, it must yield a true statement for each replacement of the variable.

4

U = N.

$a + 0 = a$.

O Look at Display 5. We will decide if each of the conditions that is expressed here is a property. Since subtraction is related to addition, we will think of what each condition means in terms of addition. The condition expressed by sentence A means to find replacements for a that satisfy $a = 0 + a$. In other words, what natural number can we add to 0 to obtain the natural number itself? What property did you use to obtain your answer?

5

U = N.

A $a - a = 0$.

B $a - 0 = a$.

C $0 - a = a$.

P Is the condition expressed by sentence A a property? Explain your answer.

Q Now look at sentence B in Display 5. The condition $a - 0 = a$ means to find replacements for a that satisfy $a = a + 0$. Is $a - 0 = a$ a property?

R Look at sentence C. The condition $0 - a = a$ means to find replacements for a that satisfy $0 = a + a$. Does $0 = a + a$ yield a true statement for each replacement of a? Is $0 - a = a$ a property?

You have learned that $0 + a = a$ is the identity-element property for addition of natural numbers. From your answers for Exercise R, you can see that $0 - a = a$ is *not* a property, since it yields a true statement only when a is replaced by 0. Since $0 - a = a$ is not a property, the number 0 is not an identity element for subtraction of natural numbers.

S Although 0 is not an identity element for subtraction, do you think that the set of natural numbers contains a number that is an identity element for subtraction?

Now let us see if there is a number that is an identity element for multiplication of natural numbers.

A Look at Display 6. What is the natural number associated with set M? With set N?

6

M = {6}.

N = {7, 14, 21}.

B Tabulate $M \times N$. Is the product $1 \cdot 3$ associated with $M \times N$? How do you know?

C Now look at Display 7. Is set N equivalent to $M \times N$? Explain your answer.

7

N = {7, 14, 21}.

M × N = {(6,7), (6,14), (6,21)}.

D Is $1 \cdot 3 = 3$ a true statement? How do you know?

E In Display 8, what is the natural number associated with set J? With set K? With set L?

8

J = {2}.

K = {5, 10, 15, 20}.

L = {1, 3, 5, 7, 9, 11}.

F Tabulate $J \times K$. Is the product $1 \cdot 4$ associated with this set? How do you know?

G Tabulate $J \times L$. What product is associated with this set? How do you know?

H Is set K equivalent to $J \times K$? Explain your answer. Is $1 \cdot 4 = 4$ a true statement? How do you know?

I Is set L equivalent to $J \times L$? Is $1 \cdot 6 = 6$ a true statement? Explain your answer.

In Exercise C you saw that $M \times N$ is equivalent to set N. In Exercise H you noted that $J \times K$ is equivalent to set K; in Exercise I you noted that $J \times L$ is equivalent to set L.

J Suppose that B is any set associated with the number 1, and that A is any set associated with a natural number. Will $B \times A$ be equivalent to set A? Explain your answer.

K You have noticed that $1 \cdot 3 = 3$, $1 \cdot 4 = 4$, and $1 \cdot 6 = 6$ are true

statements. Now suppose that B and A are any two sets associated with natural numbers, and that $1 = n(B)$ and $a = n(A)$. With what set is $1 \cdot a$ associated? Do both $1 \cdot a$ and a represent the number of members in equivalent sets? Do $1 \cdot a$ and a represent the same number?

Now we know that the product of 1 and any natural number is identical with the natural number. For this reason, we call the number 1 the *identity element for multiplication.*

The fact that the product of 1 and any natural number is the natural number itself is a property of natural numbers. This property is known as the *identity-element property for multiplication of natural numbers.* This property may be expressed as follows:

The universe for a is N. For each replacement of a, a true statement is obtained from $1 \cdot a = a$.

L Do you think that some other member of N besides 1 is also an identity element for multiplication? Explain your answer.

M Look at Display 9. For each statement expressed, tell what replacement has been made for the variable in $1 \cdot a = a$. How do you know that each sentence expresses a true statement?

9

U = N.

$1 \cdot a = a$.

A $1 \cdot 16 = 16$.

B $1 \cdot 43 = 43$.

C $1 \cdot 1 = 1$.

N Is the condition that is expressed in Display 10 a property? Explain your answer.

10

U = N.

$a \cdot 1 = a$.

Various symbols may be used to indicate division. You may have seen "3 divided by 4" written in any or all of the following ways: $3 \div 4$, $\frac{3}{4}$, $3/4$. In this lesson we will use both horizontal and diagonal lines to indicate division. In Display 11 below, for example, " $\frac{a}{a}$ " means "a divided by a." We could have written "a/a" to mean "$a \div a$."

Because you have often seen fraction numerals and ratios written with diagonal or horizontal lines, you will have to examine a situation carefully before you decide whether these lines are being used in fraction numerals, ratios, or symbols for division.

O Let us decide if each of the conditions that is expressed in Display 11 is a property. Since division is related to multiplication, we will think of what each condition means in terms of multiplication. Notice that the universe for a in these conditions does not include zero. Later on in this lesson we will explain why we have not included zero.

11

U = {1, 2, 3, ···}.

A $\frac{a}{a} = 1$.

B $\frac{a}{1} = a$.

C $\frac{1}{a} = a$.

P The condition expressed by sentence A means to find replacements for a that satisfy $a = 1 \cdot a$. What natural number can we multiply by 1 to obtain the natural

number? What property did you use to obtain your answer?

Q Is the condition that is expressed by sentence A a property?

R Now look at sentence B in Display 11. The condition $a/1 = a$ means to find replacements for a that satisfy $a = a \cdot 1$. How do you know that $a/1 = a$ is a property?

S Look at sentence C. The condition $1/a = a$ means to find replacements for a that satisfy $1 = a \cdot a$. Does $1 = a \cdot a$ yield a true statement for each replacement of a? Is $1/a = a$ a property?

You have learned that $1 \cdot a = a$ is the identity-element property for multiplication of natural numbers. From your answers for Exercise S, you can see that $1/a = a$ is *not* a property, since it yields a true statement only when a is replaced by 1. Since $1/a = a$ is not a property, the number 1 is not an identity element for division.

T Now you know that 1 is not an identity element for division. Do you think that the set of natural numbers contains a number that is an identity element for division?

Now we will decide if some other sets contain these identity elements.

A Describe each set tabulated in Display 12.

12

$K = \{0, 2, 4, \cdots\}$.

$L = \{1, 3, 5, \cdots\}$.

B Does the set of even numbers contain the identity element for addition? For multiplication? Explain your answers.

C Does the set of odd numbers contain the identity element for addition? For multiplication? Explain your answers.

In Lesson 11 you learned that set M, tabulated in Display 13, is a number system relative to addition and multiplication. In this lesson you learned that if you add any number to the number zero, you obtain the number that you added. You also learned that if you multiply any number by 1, you obtain the number that you multiplied. Now we will decide if set M has an identity element for addition and an identity element for multiplication. We will use the tables in Display 13 to help us decide.

13

$M = \{E, O\}$.

+	E	O
E	E	O
O	O	E

×	E	O
E	E	E
O	E	O

D Look at Display 14. How do you know that each sentence expresses a true statement?

14

$M = \{E, O\}$.

A $E + E = E$.

B $E + O = O$.

C $O + E = O$.

D $O + O = E$.

E $O \cdot O = O$.

F $O \cdot E = E$.

G $E \cdot O = E$.

H $E \cdot E = E$.

E Sentence A tells you that when you add E to E, you obtain E. Sentence B tells you that when you add O to E, you obtain O. When you add any member of M to E, do you obtain the member of M that you added?

F What does sentence C tell you? What does sentence D tell you?

G Why can member 0 not be an identity element for addition of members of M?

Since the sum of E and any member of M is the member itself, E is the *identity element for addition of members of M.*

H Sentence E tells you that when you multiply 0 by 0, you obtain 0. What do sentences F, G, and H tell you?

I Is the product of member 0 and any member of M the member itself? Explain your answer.

0 is the *identity element for multiplication of members of M.*

J Why can E not be an identity element for multiplication of members of M?

We have seen that if a set has an identity element for an operation, the result of performing the operation on the identity element and any member of the set is that member of the set.

Now we will study the role of the number zero in multiplication of natural numbers.

A Look at Display 15. What is the natural number associated with set E? With set F?

15

E = { }.

F = {6, 12}.

You know that to find $E \times F$ we must pair each member of set E with each member of set F.

B Since set E has no members, how many ordered pairs will there be in $E \times F$?

C Is $E \times F$ the same set as set E, tabulated in Display 15? How do you know?

D What natural number is associated with the set that has no members? Is the product $0 \cdot 2$ associated with $E \times F$? How do you know?

E Is $0 \cdot 2 = 0$ a true statement? How do you know?

F Suppose that A is any set associated with a natural number. Will $\{\} \times A$ be the same as $\{\}$? Explain your answer.

G You noticed that $0 \cdot 2 = 0$ is a true statement. You also know that $0 = n\{\}$ is a true statement. Now suppose that A is any set associated with a natural number, and that $a = n(A)$. With what set is $0 \cdot a$ associated? Does $0 \cdot a$ represent the number of members in the empty set?

Now we know that the product of zero and any natural number is the number 0. This property is the *zero property for multiplication of natural numbers.* This property may be expressed as follows:

The universe for a is N. For each replacement of a, a true statement is obtained from $0 \cdot a = 0$.

H Look at Display 16. For each statement expressed, tell what replacement has been made for the variable a in $0 \cdot a = 0$. How do you know that each sentence expresses a true statement?

16

U = N.

$0 \cdot a = 0$.

A $0 \cdot 17 = 0$.

B $0 \cdot 98 = 0$.

C $0 \cdot 0 = 0$.

I How do you know that the condition expressed in Display 17 is a property?

17

$U = N.$

$a \cdot 0 = 0.$

The properties $0 \cdot a = 0$ and $a \cdot 0 = 0$ involve a very important mathematical idea. Notice that if the product of two numbers is zero, then at least one of those numbers is zero.

J Look at Display 18. We will decide if $0/a = 0$ yields a true statement for each replacement of a from $\{1, 2, 3, \cdots\}$. You know that $0/a = 0$ means to find replacements for a that satisfy $0 = 0 \cdot a$. Does $0 = 0 \cdot a$ yield a true statement for each replacement of a from $\{1, 2, 3, \cdots\}$? Explain your answer. Does $0/a = 0$ yield a true statement for each replacement of a from $\{1, 2, 3, \cdots\}$? Explain your answer.

18

$U = \{1, 2, 3, \cdots\}.$

$\frac{0}{a} = 0.$

K For each statement expressed in Display 19, tell what replacement has been made for the variable a in $0/a = 0$. How do you know that each sentence expresses a true statement?

19

$U = \{1, 2, 3, \cdots\}.$

$\frac{0}{a} = 0.$

A $\frac{0}{14} = 0.$

B $\frac{0}{97} = 0.$

C $\frac{0}{146} = 0.$

L If you divide 0 by any natural number, what number do you obtain? Explain your answer.

We have seen that $0/a = 0$ yields a true statement for each replacement of a from $\{1, 2, 3, \cdots\}$. Now we will decide what is meant by $a/0$ for each replacement of a from $\{1, 2, 3, \cdots\}$. In other words, we will study the meaning of $1/0$, $2/0$, $3/0$, $4/0$, $\cdots$.

M Look at Display 20. What is the universe for a in $a/0$?

20

$U = \{1, 2, 3, \cdots\}.$

$\frac{a}{0}$

Suppose we let the quotient $a/0$ equal q. Then we know that the condition $a/0 = q$ means $a = q \cdot 0$. It is clear that there is no number that, when multiplied by zero, will give a number other than zero. Remember that each replacement for a is different from zero. Since we cannot find a replacement for q that satisfies $a/0 = q$, we say that $a/0$ is meaningless.

Now suppose we replace a by 0 in $a/0$. Then we obtain $0/0$. Let $0/0 = q$. $0/0 = q$ means $0 = q \cdot 0$. But, $0 = q \cdot 0$ yields a true statement for *any* replacement of q. For example, if we replace q by 8 in $0 = q \cdot 0$, then $0 = 8 \cdot 0$ is a true statement. If we replace q by 5 in $0 = q \cdot 0$, then $0 = 5 \cdot 0$ is also a true statement. Therefore, the quotient $0/0$ can be any number. Since we cannot determine which number, we say that $0/0$ is *indeterminate*.

To emphasize that $a/0$ is meaningless when a is replaced by any number except

zero, and also to emphasize that *0/0* is indeterminate, we say that ***division by zero is impossible.***

On Your Own

For each of Exercises 1 through 16, simplify the expression.

1 $16 - 0$

2 $3(42 + 0)$

3 $4 \cdot 1 + 8 \cdot 1$

4 $0 \div 9 + 3$

5 $(2 \cdot 0)5$

6 $1(49 + 108)$

7 $14 + 0 + 19$

8 $(35 + 1)2$

9 $13(6 \cdot 1)$

10 $17 \cdot 1 + 5$

11 $13 + 4 - 17$

12 $(6 \cdot 0) - 0$

13 $(15 + 0) \div 1$

14 $(8 \cdot 3) \div (6 \cdot 4)$

15 $1 \div 1$

16 $(14 + 8)0$

For each of the conditions in Exercises 17 through 30, tabulate the solution set. The universe for the variables is N.

17 $14(2x) = 28.$

18 $0 \div 13 + 5 = m.$

19 $2(16 + k) = 32.$

20 $8(g) + 8(4) = 4(6 + 4).$

21 $(26 + b) + 13 = 26 + (b + 13).$

22 $(16 \cdot 4)n = 9 \cdot 6 + 10.$

23 $(14 + 0) + 35 < 14 + (y + 35).$

24 $3(25 - x) = 25 \cdot 3.$

25 $0 \div 19 + 7 = 0 \div 36 + a.$

26 $(x \cdot 21)2 < 1(21 \cdot 2).$

27 $x(3 \cdot 12) = 9 \cdot 4.$

28 $9x \div 9 = 9.$

29 $(6 \cdot 3) - (9 \cdot 2) = x.$

30 $(42 + 5)x = 0.$

31 Set F, tabulated in Display 21, is a number system relative to the operations of addition and multiplication. Is 0 the identity element for addition? Using the tables in Display 21, write three sentences that explain your answer.

21

$F = \{0, 1, 2\}.$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

32 Is 1 the identity element for multiplication? Write three sentences that explain your answer.

EXPLORING IDEAS
Lesson 13
Booklet 7

In this lesson you will learn about composite numbers and prime numbers.

In learning about composite numbers and prime numbers in this lesson, you will use $\{1, 2, 3, \cdots\}$ as the universe. In other words, you will be working with the set of natural numbers except zero. You remember that $\{1, 2, 3, \cdots\}$ is sometimes called the set of counting numbers.

A For each statement expressed in Display 1, what replacements were made for the variables in $a \times b = c$?

1

$U = \{1, 2, 3, \cdots\}.$

$a \times b = c.$

A $3 \times 2 = 6.$

B $6 \times 1 = 6.$

You have learned that when a true statement is obtained from $a \times b = c$, the replacement for a is a factor of the replacement for c. The replacement for b

2

1	2	3	~~4~~ $_2$	5	~~6~~ $_2$	7	~~8~~ $_2$	9	~~10~~ $_2$
11	~~12~~ $_2$	13	~~14~~ $_2$	15	~~16~~ $_2$	17	~~18~~ $_2$	19	~~20~~ $_2$
21	~~22~~ $_2$	23	~~24~~ $_2$	25	~~26~~ $_2$	27	~~28~~ $_2$	29	~~30~~ $_2$

is also a factor of the replacement for *c*.

B How do you know that 2 is a factor of 6? How do you know that 6 is a factor of 6?

C Is {*1, 2, 3, 6*} the complete set of factors of 6? Explain your answer.

D Tabulate the set of factors of 8; the set of factors of 12; the set of factors of 13.

By using factors, we can learn more about the counting numbers.

E Is 1 a factor of each member of {*1, 2, 3,* ···}? Explain your answer.

F Is 2 a factor of 2? How do you know?

Now we will find other numbers that have 2 as a factor. Look at Display 2 at the top of this page.

G In Display 2 the numeral 4 has been crossed out. Is 2 a factor of 2×2, or 4?

H Name the next number after 4 that has 2 as a factor. Is the numeral for this number crossed out?

I Have we crossed out a name for the number 2×4, or 8? For the number 2×5, or 10? For the number 2×6, or 12? Is 2 a factor of each of these numbers? Numbers that have 2 as a factor are *even* numbers.

J Tabulate the set of even numbers less than 12. Is 2 a member of this set?

K Counting numbers that are not even numbers are *odd* numbers. Do odd numbers have 2 as a factor? Tabulate the set of odd numbers less than 12.

Look again at Display 2. To show that a number greater than 2 has 2 as a factor, the numeral 2 has been written below the name of that number. Notice that the name of the number, 2, we are considering is not crossed out.

L Beginning with the numeral 4 in Display 2, is every second numeral crossed out? Beginning with *any* even number in {*1, 2, 3,* ···}, is every second number an even number?

M Now we will consider the number 3. Is 3 a factor of 3? Of 3×2, or 6? Of 3×3, or 9? Of 3×4, or 12? How do you know?

N Display 3 includes the work in Display 2. In Display 3, every third numeral, beginning with the numeral 6, has been crossed out. Is each of these numerals a name for a number that has 3 as a factor?

3

1	2	3	~~4~~ $_2$	5	~~6~~ $_{2,3}$	7	~~8~~ $_2$	~~9~~ $_3$	~~10~~ $_2$
11	~~12~~ $_{2,3}$	13	~~14~~ $_2$	~~15~~ $_3$	~~16~~ $_2$	17	~~18~~ $_{2,3}$	19	~~20~~ $_2$
~~21~~ $_3$	~~22~~ $_2$	23	~~24~~ $_{2,3}$	25	~~26~~ $_2$	~~27~~ $_3$	~~28~~ $_2$	29	~~30~~ $_{2,3}$

O Notice that the numeral 6 has both 2 and 3 written below it. Why is this? Which other numerals in Display 3 are names for numbers having both 2 and 3 as factors?

The type of chart shown in Displays 2 and 3 is supposed to have been invented by a Greek scientist and mathematician named Eratosthenes, who lived over two thousand years ago. The chart is called the "Sieve of Eratosthenes." It is a "sieve" because the idea of factor is used to sift out certain numbers.

P Display 4 shows part of the Sieve of Eratosthenes for the numbers from 1 through 12. The numeral 4 is crossed out because the number 4 has 2 as a factor. Are 1 and 4 also factors of 4? Tabulate the set of factors of 4.

4

1 2 3 ~~4~~$_2$ 5 ~~6~~$_{2,3}$ 7

~~8~~$_{2,4}$ ~~9~~$_3$ ~~10~~$_{2,5}$ 11 ~~12~~$_{2,3,4,6}$

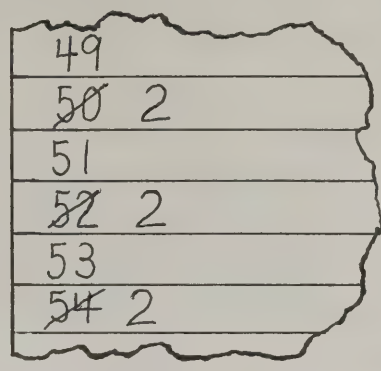
Q Tabulate the set of factors of each of the other numbers whose names are crossed out in Display 4.

R How do you know that each number whose name is crossed out must have 1 as a factor and itself as a factor?

Reminder: You have learned that a true statement is obtained from $1 \cdot a = a$ for each replacement of the variable. $U = N$. (See Lesson 12, page 63.)

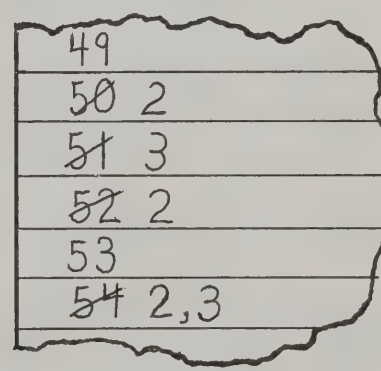
A Make a chart of the names of the natural numbers from 1 through 100. Write the numerals in vertical columns so that you will have enough space for your work. First, cross out the names for any numbers except 2 that have 2 as a factor. Write the numeral 2 next to each of these names. Display 5 shows how part of your chart might look.

5



B Now write the numeral 3 after each name of a number except 3 that has 3 as a factor. Be sure to cross out each numeral you use that has not already been crossed out. Part of your chart would now look like this:

6



C Use your work so far to help you tabulate the set whose members are the first five natural numbers having both 2 and 3 as factors. What other number besides 1 is a factor of each member of this set?

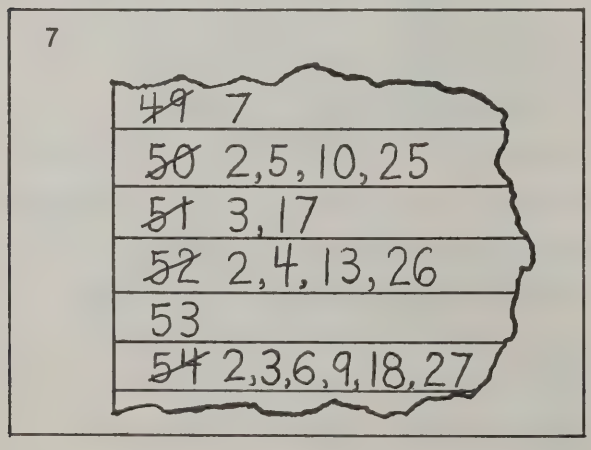
D Now consider the number 4. Is the numeral 4 crossed out in your list? Why? Have you already crossed out the names for each of the other numbers that have 4 as a factor? Why? Write the

numeral 4 after the names of numbers except 4 that have 4 as a factor.

E Is the numeral 5 crossed out in your list? Write the numeral 5 after each name for a number except 5 that has 5 as a factor. Be sure to cross out each numeral you use that has not already been crossed out.

F Have you already crossed out the numeral 6? How could you know without looking that the numerals for all the other numbers having 6 as a factor are already crossed out? Write the numeral 6 after each of these names.

G Continue to work in this manner with 7, 8, 9, and any other factors you need. Work until you are sure you have crossed out all numerals except those whose numbers have no other factors except 1 and the numbers themselves. Why will you not cross out the numeral 13, for example? Be sure to write the names of *all* the factors of the number opposite each crossed-out numeral. After you have finished, part of your work might look like this:



? **H** How do you know that each number whose name is crossed out in your list must have more than two factors?

Numbers that have more than two factors are *composite* numbers.

I Is 4 a composite number? How do you know? Tabulate the set of composite numbers that are less than 20.

> **composite number** (kəm poz'it). A member of {*1, 2, 3, ···*} that has more than two factors. The factors of 8 are 1, 2, 4, and 8; therefore, 8 is a composite number. The factors of 5 are 1 and 5; 5 is not a composite number.

J Tabulate the set of composite numbers from 20 through 30. Is this set a subset of the set of even numbers? Explain your answer.

K Is each numeral that is crossed out in your chart the name of a composite number? Explain your answer.

L Is each numeral that is not crossed out the name of a composite number? Explain your answer.

? **M** Tabulate the set of numbers from 1 through 10 that have exactly two factors. Is 1 a member of this set? Explain your answer.

N Numbers that have exactly two factors are *prime* numbers. Is 1 a prime number? Is 2 a prime number? Is 3 a prime number? Explain your answers.

> **prime number** (prīm). A member of {*1, 2, 3, ···*} that has exactly two factors. The only factors of 7 are 1 and 7; therefore, 7 is a prime number.

O Tabulate the set of even numbers from 1 through 10. Which members of this set are prime numbers?

P Is 12 a prime number? Is 14 a prime number? Is any even number greater

than 2 a prime number? Explain your answer.

Q Look at Display 8. Is each member of set S a prime number? Is S the set of all prime numbers? Explain your answers.

8

$S = \{3, 7, 11, 13, 17, 19\}$.

R For each member of S, tabulate the set of factors of that member. Is each prime number a factor of itself? What is the other factor of a prime number?

S The square of 5 is 5×5, or 25. Tabulate the set of factors of 25. The square of 7 is 7×7, or 49. Tabulate the set of factors of 49. If a number is a square of a prime number, the square has how many factors? How do you know?

T Is there a member of $\{1, 2, 3, \cdots\}$ whose set of factors is the empty set? What is the least number of factors that a member of $\{1, 2, 3, \cdots\}$ can have?

U Is there a member of $\{1, 2, 3, \cdots\}$ whose set of factors is an infinite set? Explain your answer.

V Is the union of the set of primes and one set of composites equal to the set of counting numbers? Explain your answer.

A Now look at Display 9. The display includes the names of the prime numbers that are less than 60. The prime numbers from 1 through 10 are named in the first row; the prime numbers from 11 through 20 are named in the second row; and so on. Copy and complete the chart in Display 9 so that it includes the names of all prime numbers from 1 through 100. Use the chart you made to help in doing this.

9

1 through 10		2	3	5	7	
11 through 20	11		13		17	19
21 through 30			23			29
31 through 40	31				37	
41 through 50	41		43		47	
51 through 60			53			59

B Now consider each member of $\{1, 2, 3, \cdots\}$, not just the members less than 101. Is it likely that every ten members of this set (the numbers from 101 through 110, from 111 through 120, and so on) will include the same number of prime numbers?

C Do you think that there is a greatest prime number? Explain your answer.

D Some prime numbers have only one other natural number between them. What is the first pair like this shown in Display 9?

Two prime numbers are called *twin* primes if their difference is two. There is just one natural number between twin primes.

E What other twin primes are named in Display 9 besides 3 and 5?

F Look at the chart that you made for the numbers from 1 through 100. What twin primes are greater than 60 and less than 101?

Mathematicians have wondered whether there is a largest pair of primes that are twin primes. No one knows the answer to this question, but there is good evidence to support the belief that the set of twin primes is an infinite set.

G Now look at Display 10 on page 72. Is the number 8 expressed as the sum of two

prime numbers? Is each of the other even numbers expressed as the sum of two prime numbers?

10

A $8 = 5 + 3$. **C** $12 = 5 + 7$.

B $10 = 3 + 7$. **D** $14 = 7 + 7$.

H Express each of the following even numbers as a sum of two prime numbers:

4 6 16 20 26 28 34 36

You can see that all the even numbers in the chart that you made can be expressed as the sum of two prime numbers. Mathematicians know that this is true of very large even numbers, but they do not know if it is true of all even numbers.

I Now look at Display 11. The universe is the set of counting numbers greater than 1. The first row of the table tells us that 3 is the only prime number between 2 and 2×2. What prime number is between 3 and 2×3? Between 4 and 2×4?

11

$U = \{2, 3, 4, \cdots\}$.

x	$2x$	Prime numbers between x and $2x$
2	4	3
3	6	5
4	8	5, 7
⋮	⋮	⋮
16	32	17, 19, 23, 29, 31
⋮	⋮	⋮

J What prime numbers are between 16 and 2×16? What prime numbers are between 30 and 2×30?

Mathematicians have wondered whether there is always at least one prime number between a number greater than 1, and two times that number. They have been able to prove that no matter how great a number is selected, there is always at least one prime number between the selected number and two times that number.

On Your Own

Tabulate the set of factors of each number named in Exercises 1 through 8.

1 27 **3** 29 **5** 57 **7** 167

2 3 **4** 32 **6** 120 **8** 227

9 Which of the numbers named in Exercises 1 through 8 are prime numbers? Which are composite numbers?

10 $\{1, 19\}$ is the set of factors of what number?

11 A is the set of factors of any prime number. How many members does A have?

12 In this lesson you learned about twin primes. Are there any numbers that could be called triple primes? If so, name them. Three primes are triple primes if there is just one natural number between the first and second and between the second and third.

13 K is the set of factors of 16; L is the set of factors of 4. Is one set a subset of the other? Explain your answer.

14 How many even numbers have just two factors? Name these even numbers.

Each numeral in Exercises 15 through 18 is written in a base other than ten. Use a base-ten numeral to name the number that each numeral expresses. Then tell whether the number is odd or even.

15 11 (base three) **17** 33 (base five)

16 12 (base five) **18** 101 (base two)

19 12 expresses an even number; the ones' digit is the numeral 2. What other symbols can be the ones' digit of a base-ten numeral for an even number? Give an example for each answer.

20 101_{three} expresses an even number; the ones' digit is 1. What other symbols could be the ones' digit of a base-three numeral for an even number? Give an example for each answer.

For each number named in Exercises 21 through 28, tell whether that number is even or odd. Try to answer these exercises without simplifying the expressions.

21 2×5 **22** $3 + 7$ **23** $6 \times 5 \times 3$ **24** $2 + 16$ **25** $7 + 8$ **26** $3(4 + 7)$ **27** $128 - 37$ **28** $5(9 + 13)$

29 Copy the following table and complete it for natural numbers from 1 through 30.

Number	Factors	Number of factors	Sum of factors
1	1	1	1
2	1,2	2	3
3	1,3	2	4
4	1,2,4	3	7
5	1,5	2	6
6	1,2,3,6	4	12
7	1,7	2	8
8	1,2,4,8	4	15

Use the table you made for Exercise 29 to help you answer Exercises 30 through 32.

30 Which numbers have exactly two factors?

31 Which numbers have exactly three factors?

32 If the universe for n is $\{1, 2, 3, \cdots, 30\}$, for what numbers will $2n$ become the sum of the factors of n? These numbers are *perfect* numbers. It is not known how many perfect numbers there are or whether there are any odd perfect numbers.

33 In this lesson you made a chart of the names of numbers from 1 through 100 and their factors. Use this chart to help tabulate the set of perfect numbers from 1 through 100.

34 Is 2 a factor of each of the components of $(12,24)$? Explain your answer. What other numbers are factors of each of the components of $(12,24)$?

For each ordered pair named in Exercises 35 through 40, name two numbers each of which is a factor of both the first component and the second component.

35 (7,14) **36** (6,8) **37** (100,200) **38** (10,16) **39** (45,21) **40** (26,130)

41 If the replacement for x is a factor of the replacement for y, and the replacements are not equal, from which of these conditions will you obtain a true statement?

a $x < y$. **b** $x > y$.

For each number named in Exercises 42 through 49 tell what prime numbers are factors of that number.

42 84 **43** 89 **44** 112 **45** 286 **46** 10,000 **47** 3000 **48** 840 **49** 660

For each ordered pair named in Exercises 50 through 53, find the greatest number that is a factor of each of the components of that pair.

50 (30,60) **51** (7,11) **52** (70,130) **53** (98,1)

EXPLORING IDEAS
Lesson 14
Booklet 7

In this lesson you will learn about the prime factors of a number.

A Look at Display 1. First 24 is expressed as the product of 4 and 6. Are 4 and 6 prime numbers? What prime numbers can be multiplied to obtain 4? To obtain 6?

1

$24 = 4 \times 6.$

$4 = 2 \times 2.$

$6 = 2 \times 3.$

$24 = 2 \times 2 \times 2 \times 3.$

$24 = 2^3 \times 3.$

B If we think of 4 as 2×2, and 6 as 2×3, then we think of 24 as 4×6, or $2 \times 2 \times 2 \times 3$. Are 2 and 3 prime numbers? Have we expressed 24 as a product of prime numbers?

C Are $2 \times 2 \times 2 \times 3$ and $2^3 \times 3$ the same number? Explain your answer.

Reminder: Numbers like 2^3 are powers. The base of 2^3 is 2, and the exponent of 2^3 is 3. The exponent tells us how many times the base is used as a factor. In 2^3, 2 is used as a factor three times. So $2^3 = 2 \times 2 \times 2$ is a true statement. (See page 33, BOOKLET 5.)

D Now look at Display 2. In 20×5, which factor is not a prime number? How was $2 \times 10 \times 5$ obtained from 20×5?

2

$100 = 20 \times 5.$

$100 = 2 \times 10 \times 5.$

$100 = 2 \times 2 \times 5 \times 5.$

$100 = 2^2 \times 5^2.$

E In $2 \times 10 \times 5$, which factor is not a prime number? How was $2 \times 2 \times 5 \times 5$ obtained from $2 \times 10 \times 5$?

F When we think of 100 as $2 \times 2 \times 5 \times 5$, are we thinking of 100 as a product of prime numbers? Is $2^2 \times 5^2$ the same number as $2 \times 2 \times 5 \times 5$? How do you know?

G Now look at Display 3. In 10×35, is either factor a prime number?

3

$350 = 10 \times 35.$

$350 = 2 \times 5 \times 35.$

H How was $2 \times 5 \times 35$ obtained from 10×35? In $2 \times 5 \times 35$, which factor is not a prime number? This factor can be expressed as the product of what prime numbers?

I When we write $2 \times 5 \times 5 \times 7$, have we expressed 350 as a product of prime numbers? Use power notation to express $2 \times 5 \times 5 \times 7$.

Reminder: When we write 10^2 to express the number 100, we are using power notation. By using power notation, we can express the number 18, or $2 \times 3 \times 3$, by the numeral 2×3^2. (See page 33, BOOKLET 5.)

You have expressed the number 24 as $2 \times 2 \times 2 \times 3$, or $2^3 \times 3$. You have expressed the number 100 as $2 \times 2 \times 5 \times 5$, or $2^2 \times 5^2$. You have expressed the number 350 as $2 \times 5 \times 5 \times 7$, or $2 \times 5^2 \times 7$. In each case you have thought of a number as a product of prime numbers. Mathematicians have shown that any composite number can be thought of as a product of prime numbers. For example, $6 = 2 \times 3$, $8 = 2^3$, $9 = 3^2$, $10 = 2 \times 5$, and so on. When we express a number as a product of prime numbers, we say that we have found a *complete factorization* of the number.

Complete factorization (fak'tər ə zā'shən). If a composite number is expressed as a product of prime numbers, the product is a complete factorization of the given number. 3×5 is a complete factorization of 15. Both $3 \times 3 \times 2$ and $3^2 \times 2$ are complete factorizations of 18.

Mathematicians think of a prime number as a complete factorization of itself. 2 is the complete factorization of 2, 3 is the complete factorization of 3, and so on for each prime number.

J Which numbers in Display 4 are expressed as products of prime numbers?

4

$9 = 3 \times 3.$	$120 = 4 \times 30.$
$12 = 4 \times 3.$	$70 = 5 \times 7 \times 2.$
$16 = 4 \times 4.$	$144 = 2 \times 72.$

K If we divide 12 by 4 we obtain 3; 3 and 4 are factors of 12. But 4 is not a prime number. So we continue to divide. If we divide 4 by the prime number 2, we obtain 2. Thus 2 is a factor of 4 and also of 12. Is $2 \times 2 \times 3$, or $2^2 \times 3$, a complete factorization of 12?

L Find a complete factorization of each of the numbers expressed in Display 4 that is not already expressed as a product of prime numbers. Use power notation in your answers.

Notice that we can use division to help us find the factors of a number. A factor of a number divides the number without a remainder. For example, 4 is a factor of 12 because it divides 12 without a remainder. $12/4 = 3$. We say in this example that 12 is divisible by 4. Each number is divisible by any of its factors.

We can also use division to test whether a number is a factor of another number. 17 is not divisible by 5. So 5 is not a factor of 17.

M How do you know that 144 is divisible by 36?

N Is 14 a factor of 52? Explain your answer.

O Find a complete factorization of each member of $\{28, 32, 45, 75, 98, 121\}$. Use power notation to express your answers.

P Is $2 \times 9 \times 7$ a complete factorization of 126? Explain your answer.

Q Simplify each of the following expressions: $2^2 \times 3 \times 2$, $2^3 \times 3$, $3 \times 2 \times 2 \times 2$.

R Does each expression in Exercise Q represent a complete factorization of the number 24?

Mathematicians have shown that any counting number greater than 1 can be expressed as a product of prime numbers in just one way, but the numbers may be used in any order. For example, 2, 5, and 3 are the only prime factors of 30. There is only one complete factorization of 30, but you may use the numbers 2, 5, and 3 in any order you wish. This property of counting numbers greater than 1 is expressed below.

Every counting number greater than 1 can be factored into primes in only one way, except for the order in which the primes occur in the product.

This property is the *unique factorization* property of the counting numbers. The word *unique* as used here means that there is just one way to express a number as a product of prime numbers.

S Now look at Display 5 on the next page. 6 is expressed as a product

in how many different ways? Which product is a complete factorization of 6?

5

A $6 = 2 \times 3$.

B $6 = 2 \times 3 \times 1$.

C $6 = 2 \times 3 \times 1 \times 1$.

? **T** Suppose that the number 1 is considered to be a prime number. How many different complete factorizations of 6 would there be? Explain your answer.

? **U** If we considered 1 as a prime number, would there be a unique factorization property of counting numbers? Explain your answer.

V Find a complete factorization of 18. Find a complete factorization of 45.

W How do you know that 18 is an even number? How do you know that 45 is an odd number?

Now you will relate the factors of one number with the factors of another number.

A Look at Display 6. Which set is the set of factors of 8? Which set is the set of factors of 20?

6

A = {1, 2, 4, 8}.

B = {1, 2, 4, 5, 10, 20}.

B Is 2 a factor of 8? Is 2 also a factor of 20? When a number is a factor of two other numbers, it is a *common* factor.

C Is 2 a common factor of 8 and 20? Is 4 a common factor of 8 and 20? How do you know?

D Tabulate the set of common factors of 8 and 20. Is this set a subset of the set of factors of 8? Is this set a subset of the set of factors of 20?

E Tabulate the set of factors of 12. Tabulate the set of factors of 30. Tabulate the set of common factors of 12 and 30.

F Tabulate the set of common factors of 5 and 3. Of 5 and 7. Of 7 and 3.

G How do you know that {*1*} is the set of common factors of any pair of prime numbers?

H Tabulate the set of common factors of 25 and 30. Tabulate the set of common factors of 15 and 27.

I Is {*1, 2, 4*} the set of common factors of 12 and 16? What is the greatest member of {*1, 2, 4*}? This number is the *greatest common factor* of 12 and 16.

Greatest common factor.
The greatest common factor of two counting numbers is the greatest number that is a factor of both of them. 15 is the greatest common factor of 30 and 45.

J Is {*1, 2, 7, 14*} the set of common factors of 14 and 28? What is the greatest common factor of 14 and 28?

K Tabulate the set of common factors of 26 and 130. Can a number greater than 26 be a common factor of 26 and 130?

L To find the common factors of 100 and 210, you would first find all the factors of 100 and all the factors of 210. How would you determine which numbers are the common factors of 100 and 210? How would you determine which number is the greatest common factor?

M Look at Display 7 on the next page. Tabulate the set of common factors of 100 and 210. What number is the greatest common factor?

7

100 = {1, 2, 4, 5, 10, 20, 25, 50, 100}.

210 = {1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210}.

When numbers have many factors, it is inconvenient to tabulate the set of factors of each number. There is an easier method of determining the greatest common factor of two numbers.

N Look at Display 8. Is 100 expressed as a product of prime numbers? Is 210 expressed as a product of prime numbers?

8

$100 = 2 \times 5 \times 2 \times 5.$

$210 = 2 \times 5 \times 3 \times 7.$

O What prime numbers are factors of both 100 and 210? What is the product of these prime factors? Is this product the greatest common factor of 100 and 210?

P Find a complete factorization of 300. Of 550.

Q What prime numbers are factors of both 300 and 550?

R What is the product of the common prime factors of 300 and 550? Is the product the greatest common factor of 300 and 550? Is the product of 2 and 5^2 the greatest common factor?

You have discovered that 2 and 5 are the common prime factors of 300 and 550. But 5^2 is also a factor of both 300 and 550. We therefore use 5^2 to find the greatest common factor. 2^2 is a factor of 300 but not of 550. 2^2 cannot be used to find the greatest common factor. We always use the greatest power of a common prime factor that is found in both numbers. 5^2 and 2 are the greatest powers of 5 and 2 to be found in both 300 and 550.

A Look at Display 9. Are 2 and 3 the only common prime factors of 24, 36, and 120?

9

$24 = 2 \times 2 \times 2 \times 3.$

$36 = 2 \times 2 \times 3 \times 3.$

$120 = 2 \times 2 \times 2 \times 3 \times 5.$

B For each complete factorization expressed in Display 9, tell how many times 2 is used as a factor. How do you know that 2^2 is the greatest power of 2 that is a factor of 24, 36, and 120?

C For each complete factorization expressed in Display 9, tell how many times 3 is used as a factor. How do you know that 3, or 3^1, is the greatest power of 3 that is a common factor of 24, 36, and 120?

D Now you know that 2^2 and 3^1 are the greatest powers of the prime numbers 2 and 3 that are also common factors of 24, 36, and 120. What is the product of 2^2 and 3? Is $2^2 \times 3$, or 12, the greatest common factor of 24, 36, and 120?

E Find a complete factorization of 14, 21, and 28. What prime numbers are common factors of 14, 21, and 28?

F For each common prime factor of 14, 21, and 28, tell what power of the prime number is also a common factor of 14, 21, and 28. Is the product of these powers the greatest common factor of 14, 21, and 28?

G Find the greatest common factor of 100, 125, and 75.

Now you know how to find the greatest common factor of two or more numbers. You can do this in two ways. You may tabulate the set of factors of each number and then

find the greatest number common to both sets. Or you may find a complete factorization of each number. Then you find the product of powers of the common prime factors of the numbers. You use the greatest power of each prime number that is a factor of all the numbers.

On Your Own

1 Tabulate the set of factors of each number named below. You will use these sets to help you answer Exercise 2.

a 6	**d** 15	**g** 30	**j** 36
b 8	**e** 16	**h** 27	**k** 45
c 12	**f** 21	**i** 28	**l** 72

2 Using your answers for Exercise 1, tabulate the set of common factors of each pair of numbers named below.

a 6, 8	**d** 27, 15	**g** 16, 36
b 8, 12	**e** 16, 12	**h** 30, 72
c 15, 30	**f** 30, 27	**i** 15, 45

3 Name the greatest common factor of each pair of numbers named in Exercise 2.

4 Name the greatest common factor of each group of numbers named below.

a 15, 25	**e** 15, 30, 36
b 18, 30	**f** 12, 24, 28
c 24, 36	**g** 40, 48, 72
d 36, 45	**h** 20, 50, 100

5 What is the greatest common factor of 1 and 6? Of 1 and 8? Of 1 and 300? Of 1 and any member of $\{1, 2, 3, \cdots\}$?

6 Do any two members of $\{1, 2, 3, \cdots\}$ always have a common factor? If so, what is the common factor?

7 Tabulate a set of three composite numbers such that the only common factor of the numbers is 1. When the only common factor of composite numbers is 1, the composite numbers are *relatively prime.*

8 Are 9 and 10 relatively prime? Are 9 and 18 relatively prime? Explain your answers.

9 What is the greatest common factor of any two prime numbers? How do you know?

10 For each pair of numbers named below, first find a complete factorization of each number. Then name the greatest common factor of the two numbers.

a 24, 60	**e** 60, 84
b 36, 90	**f** 165, 234
c 72, 108	**g** 306, 1173
d 25, 125	**h** 2040, 2184

11 The following pairs of numbers are named by base-seven numerals. Find the greatest common factor of each pair of numbers. Express your answer as a base-seven numeral.

a 15_{seven}, 33_{seven}

b 26_{seven}, 50_{seven}

c 66_{seven}, 110_{seven}

12 Determine the least prime factor of each number named below.

a 115	**c** 321	**e** 539
b 135	**d** 484	**f** 121

13 What number is the least prime factor of any even number?

14 For each exercise below, name the least number that has both numbers as factors.

a 1, 2	**c** 3, 2	**e** 4, 12
b 7, 2	**d** 4, 11	**f** 3, 9

15 Suppose that $a*b$ means that $a*b$ is the greatest common factor of a and b.

a Is the set of counting numbers closed under the operation $*$?

b Is operation $*$ commutative for counting numbers? Give two examples to explain your answer.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 SH 70 69 68 67 66 65 64 63 62 61

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